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QUANTUM LOGARITHMIC SOBOLEV INEQUALITIES FOR QUANTUM MANY-BODY SYSTEMS: AN APPROACH VIA QUASI-FACTORIZATION OF THE RELATIVE ENTROPY

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for the degree of *Doctor in Mathematics* by

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*A mis padres, Antonio y Ángela.
Y a los suyos, Francisco, Dolores, Ángel y Manoli.*

Contents

ACKNOWLEDGEMENTS	11
NOTATION AND ACRONYMS	15
LIST OF FIGURES	21
SUMMARY	21
RESUMEN	23
 I INTRODUCTION AND PRELIMINARIES	 27
1 INTRODUCTION	29
1.1 NOTATION AND BACKGROUND	29
1.1.1 NOTATION	29
1.1.2 QUANTUM DISSIPATIVE EVOLUTIONS	30
1.1.3 MIXING TIME AND LOG-SOBOLEV INEQUALITIES	32
1.2 STRATEGY TO FIND POSITIVE LOG-SOBOLEV CONSTANTS	34
1.3 MAIN RESULTS	38
1.3.1 QUASI-FACTORIZATION OF THE RELATIVE ENTROPY	38
1.3.2 LOGARITHMIC SOBOLEV INEQUALITIES	43
1.3.3 DATA PROCESSING INEQUALITY FOR THE BS-ENTROPY	45
1.4 ORGANIZATION OF THE THESIS	48

2	INTRODUCCIÓN	51
2.1	NOTACIÓN Y ANTECEDENTES	51
2.1.1	NOTACIÓN	51
2.1.2	EVOLUCIONES DISIPATIVAS CUÁNTICAS	52
2.1.3	TIEMPO DE EQUILIBRACIÓN Y DESIGUALDADES DE LOG-SOBOLEV	54
2.2	ESTRATEGIA PARA ENCONTRAR CONSTANTES DE LOG-SOBOLEV POSITIVAS	57
2.3	RESULTADOS PRINCIPALES	61
2.3.1	QUASI-FACTORIZACIÓN DE LA ENTROPÍA RELATIVA	61
2.3.2	DESIGUALDADES LOGARÍTMICAS DE SOBOLEV	65
2.3.3	DESIGUALDAD DE PROCESAMIENTO DE DATOS PARA LA ENTROPÍA BS	67
2.4	ORGANIZACIÓN DE LA TESIS	70
3	CLASSICAL CASE	73
3.1	NOTATION AND BASIC CONCEPTS	73
3.2	GIBBS MEASURES	75
3.2.1	THE ISING MODEL	77
3.3	THE DYNAMICS	78
3.4	MIXING CONDITIONS	80
3.5	POSITIVE ENTROPY CONSTANT FROM (DPP02)	82
4	PRELIMINARIES	89
4.1	VON NEUMANN ENTROPY AND RELATIVE ENTROPY	90
4.2	NON-COMMUTATIVE \mathbb{L}_p-SPACES	94
4.3	CONDITIONAL EXPECTATIONS	96
4.4	OPERATOR CONVEX FUNCTIONS	98
4.5	LOG-SOBOLEV INEQUALITIES	100
4.6	GIBBS STATES	104
4.7	QUANTUM MARKOV CHAINS	104
II	QUASI-FACTORIZATION OF THE RELATIVE ENTROPY	107
5	SUPERADDITIVITY OF THE RELATIVE ENTROPY	111
5.1	STEP 1: ADDITIVE ERROR TERM FOR THE DIFFERENCE OF RELATIVE ENTROPIES	113
5.2	STEP 2: ERROR TERM WITH LIEB'S EXTENSION OF GOLDEN-THOMPSON	113
5.3	STEP 3: HÖLDER'S AND PINSKER'S INEQ. TO GET BACK A RELATIVE ENTROPY	116
5.4	STEP 4: NON-COMMUTATIVE \mathbb{L}_p-NORMS TO GET A NICER ERROR TERM	117
5.5	IMPLICATIONS OF THIS RESULT	121

6	QUANTUM CONDITIONAL RELATIVE ENTROPY	123
6.1	CONDITIONAL RELATIVE ENTROPY	124
6.2	A FORMULA FOR THE CONDITIONAL RELATIVE ENTROPY	125
6.3	CONDITIONAL RELATIVE ENTROPY BY EXPECTATIONS	128
6.4	COMPARISON OF DEFINITIONS	129
6.5	RELATION WITH THE CLASSICAL CASE	132
6.6	GENERAL CONDITIONAL RELATIVE ENTROPY BY EXPECTATIONS	134
7	QUASI-FACTORIZATION OF THE RELATIVE ENTROPY	137
7.1	FIRST RESULTS ON QUASI-FACTORIZATION	139
7.2	QUASI-FACTORIZATION FOR σ A TENSOR PRODUCT	143
7.3	QUASI-FACTORIZATION FOR THE CONDITIONAL RELATIVE ENTROPY	145
7.4	QUASI-FACTORIZATION FOR THE CRE BY EXPECTATIONS	146
7.4.1	STEP 1: ADDITIVE ERROR TERM FOR THE DIFFERENCE OF RELATIVE ENTROPIES	147
7.4.2	STEP 2: ERROR TERM WITH LIEB'S EXTENSION OF GOLDEN-THOMPSON	147
7.4.3	STEP 3: SPLITTING THE ERROR TERM INTO TWO PARTS	150
7.4.4	STEP 4: HÖLDER'S AND PINSKER'S INEQUALITIES TO OBTAIN A RELATIVE ENTROPY	151
8	STRONG QUASI-FACTORIZATION OF THE RELATIVE ENTROPY	155
8.1	STRONG QUASI-FACTORIZATION FOR σ A TENSOR PRODUCT	156
8.2	STRONG QUASI-FACTORIZATION FOR QUANTUM MARKOV CHAINS	157
8.3	CLUSTERING OF CORRELATIONS	159
8.4	STRONG QUASI-FACTORIZATION FOR THE GCRE BY EXPECTATIONS	164
8.4.1	STEP 1: ADDITIVE ERROR TERM FOR THE DIFFERENCE OF RELATIVE ENTROPIES	165
8.4.2	STEP 2: ERROR TERM WITH LIEB'S EXTENSION OF GOLDEN-THOMPSON	165
8.4.3	STEP 3: CONDITIONAL \mathbb{L}_1 -CLUSTERING OF CORRELATIONS TO BOUND THE ERROR TERM	167
8.4.4	STEP 4: PROPERTIES OF WEIGHTED \mathbb{L}_p -NORMS TO OBTAIN A RELATIVE ENTROPY	168
8.5	OTHER CLUSTERING CONDITIONS	169
8.5.1	CONDITIONAL \mathbb{L}_2 -CLUSTERING OF CORRELATIONS	169
8.5.2	COVARIANCE-ENTROPY CLUSTERING OF CORRELATIONS	171
III	LOGARITHMIC SOBOLEV INEQUALITIES	181
9	TENSOR PRODUCT CASE	185
9.1	LOGARITHMIC SOBOLEV INEQUALITY FOR A TENSOR PRODUCT FIXED POINT	186
10	HEAT-BATH DYNAMICS IN DIMENSION 1	193
10.1	HEAT-BATH DYNAMICS AND CONDITIONAL LOG-SOBOLEV CONSTANT	194
10.2	TECHNICAL TOOLS	195

10.3	POSITIVITY OF THE LOG-SOBOLEV CONSTANT FOR THE HEAT-BATH DYNAMICS	199
10.3.1	STEP 1: QUASI-FACTORIZATION OF THE ENTROPY INTO TWO REGIONS	200
10.3.2	STEP 2: QUASI-FACTORIZATION OF THE ENTROPY INTO MANY REGIONS	201
10.3.3	STEP 3: LOWER BOUND FOR THE LOG-SOBOLEV CONSTANT IN TERMS OF COND. ONE	201
10.3.4	STEP 4: POSITIVE CONDITIONAL LOG-SOBOLEV CONSTANT FOR THE HEAT-BATH GEN.	203
10.4	MIXING CONDITION AND STRONG QUASI-FACTORIZATION	205
10.4.1	MIXING CONDITION	205
10.4.2	STRONG QUASI-FACTORIZATION	209
10.5	EXTENSION TO A LARGER DIMENSION	211
11	DAVIES DYNAMICS	213
11.1	DAVIES GENERATORS	214
11.2	REDUCTION FROM GLOBAL TO CONDITIONAL LOG-SOBOLEV CONSTANT	218
11.3	DISCUSSION ON THE POSITIVITY OF THE LOG-SOBOLEV CONSTANT	224
11.4	EXAMPLE	225
IV	DATA PROCESSING INEQUALITY	229
12	A STRENGTHENED DATA PROCESSING INEQUALITY FOR THE BS-ENTROPY	235
12.1	STANDARD AND MAXIMAL f-DIVERGENCES	235
12.1.1	STANDARD f -DIVERGENCES	235
12.1.2	MAXIMAL f -DIVERGENCES	237
12.2	A CONDITION FOR EQUALITY	239
12.3	STRENGTHENED DATA PROCESSING INEQUALITY FOR THE BS-ENTROPY	245
12.4	ON THE DATA PROCESSING INEQUALITY FOR MAXIMAL f-DIVERGENCES	249
12.5	EXTENSION OF THE PREVIOUS RESULTS TO GENERAL QUANTUM CHANNELS	253
	CONCLUSIONS AND OPEN PROBLEMS	257
	FUTURE WORK	258
	OPEN PROBLEMS	259
	CONCLUSIONES Y PROBLEMAS ABIERTOS	261
	TRABAJO FUTURO	262
	PROBLEMAS ABIERTOS	263
	APPENDIX: NUMERICAL RADIUS ATTAINING COMPACT LINEAR OP.	267
	BIBLIOGRAPHY	271
	BOOKS	271
	ARTICLES	272

INDEX	281
Index	281



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“And, when you want something, all the universe conspires in helping you to achieve it.”
The Alchemist, Paulo Coelho.

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This picture corresponds a landscape of Madrid, the city where I have spent most of the time during my PhD.

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NOTATION AND ACRONYMS

Linear algebra

\mathcal{H}, \mathcal{K}	Finite-dimensional Hilbert spaces
$\mathcal{B}(\mathcal{H})$	Algebra of bounded linear operators on \mathcal{H}
$\mathcal{A}(\mathcal{H})$	Set of Hermitian operators on \mathcal{H}
$\mathcal{A}^+(\mathcal{H})$	Set of positive Hermitian operators on \mathcal{H}
$\mathcal{S}(\mathcal{H})$	Set of density matrices (quantum states)
$X, Y \dots$	Elements of $\mathcal{B}(\mathcal{H})$
$f, g \dots$	Observables (Hermitian operators)
$\rho, \sigma \dots$	States (density matrices)
$\rho > 0, \rho \geq 0$	Positive definite, resp. semidefinite, state
ρ^0	Support of ρ
X^*	Adjoint (or Hermitian conjugate) of an operator $X \in \mathcal{B}(\mathcal{H})$
$\mathbb{1}_{\mathcal{H}}$	Identity operator on \mathcal{H}
$\mathcal{T} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$	Superoperator, quantum channel (CPTP map)
$\mathcal{T}^* : \mathcal{B}(\mathcal{K}) \rightarrow \mathcal{B}(\mathcal{H})$	Adjoint map of $\mathcal{T} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$
$\text{Id}_{\mathcal{H}}$	Identity map on $\mathcal{B}(\mathcal{H})$
\mathcal{M}, \mathcal{N}	Matrix algebras
$\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$	Conditional expectation
$\sigma_{\mathcal{N}}, \rho_{\mathcal{N}}$	$\sigma_{\mathcal{N}} := \mathcal{E}(\sigma), \rho_{\mathcal{N}} := \mathcal{E}(\rho)$
$\sigma_{\mathcal{T}}, \rho_{\mathcal{T}}$	$\sigma_{\mathcal{T}} := \mathcal{T}(\sigma), \rho_{\mathcal{T}} := \mathcal{T}(\rho)$
L_{ρ}, R_{σ}	Left multiplication by ρ , right multiplication by σ
$\Delta_{\sigma, \rho}$	Modular operator, $\Delta_{\sigma, \rho}(\cdot) = \sigma(\cdot)\rho^{-1}$
Γ_{σ}	Gamma operator, $\Gamma_{\sigma}(\cdot) = \sigma^{1/2}(\cdot)\sigma^{1/2}$
$\Gamma, \Gamma_{\mathcal{T}}, \Gamma_{\mathcal{N}}$	$\Gamma = \sigma^{-1/2}\rho\sigma^{-1/2}, \Gamma_{\mathcal{T}} = \sigma_{\mathcal{T}}^{-1/2}\rho_{\mathcal{T}}\sigma_{\mathcal{T}}^{-1/2}, \Gamma_{\mathcal{N}} = \sigma_{\mathcal{N}}^{-1/2}\rho_{\mathcal{N}}\sigma_{\mathcal{N}}^{-1/2}$
$\text{tr}[\cdot]$	Trace
$[X, Y]$	Commutator of X and Y

This picture was taken while I was leaving Florianópolis (Brazil) after the workshop *Q-Turn: Changing paradigms in quantum science*. It is used here to represent the starting point of the entertaining trip throughout the thesis.

Quantum information theory

\mathcal{H}_A	Hilbert space associated to a quantum system A
$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$	Bipartite Hilbert space
$\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$	Tripartite Hilbert space
$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$	Multipartite Hilbert space
$\mathcal{R}_\Phi^\sigma(\cdot)$	Petz recovery map for the channel Φ with respect to σ
$\mathcal{B}_\Phi^\sigma(\cdot)$	BS-recovery condition for the channel Φ with respect to σ
$\text{tr}_A[\cdot]$	Partial trace over A
τ_A	Identity normalized in A
$ \psi\rangle_A$	Pure state on A
$ \psi\rangle\langle\psi _A$	Density operator associated to $ \psi\rangle_A$
$\mathbb{E}^*(\cdot)$	Heat-bath conditional expectation

Quantum spin lattices

$\Lambda \subset \mathbb{Z}^d$	Finite d -dimensional lattice
\mathcal{B}_Λ	Set of bounded linear operators on \mathcal{H}_Λ
\mathcal{A}_Λ	Set of Hermitian operators on \mathcal{H}_Λ
\mathcal{S}_Λ	Set of density operators on \mathcal{H}_Λ
$x \in \Lambda$	Site
$f_\Lambda, g_\Lambda, \dots$	Observables on \mathcal{H}_Λ
$\rho_\Lambda, \sigma_\Lambda, \dots$	States on \mathcal{H}_Λ
$\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \rightarrow \mathcal{S}_\Lambda$	Lindbladian (or Liouvillian)
$\mathcal{T}_t^* = e^{t\mathcal{L}_\Lambda^*}$	Quantum Markov semigroup
H_Λ	Hamiltonian on Λ
$\sigma_\Lambda = e^{-\beta H_\Lambda} / \text{tr}[e^{-\beta H_\Lambda}]$	Gibbs state
$A, B \subseteq \Lambda$	Subregions of Λ
$\partial^+ A$	Boundary of A
Φ	Potential

Non-commutative \mathbb{L}_p -spaces

$\ \cdot\ _p$	Schatten p -norm
$\ \cdot\ _1$	Trace norm
$\ \cdot\ _\infty$	Operator norm
$\langle \cdot, \cdot \rangle_{\text{HS}}$	Hilbert-Schmidt inner product
$\ \cdot\ _{p,\sigma}$	Weighted p -norm (with weight σ)
$\langle \cdot, \cdot \rangle_\sigma$	Weighted (or KMS) inner product
$\langle \cdot, \cdot \rangle_{\text{GNS},\sigma}$	GNS inner product

Quantum distances and entropies

$S(\cdot)$	Von Neumann entropy
$S_\rho(A B)$	Conditional entropy in A of ρ
$D(\cdot \cdot)$	Relative entropy
$D_A(\cdot \cdot)$	Conditional relative entropy in A
$D_A^E(\cdot \cdot)$	Conditional relative entropy by expectations in A
$D_{\mathcal{N}}^{\mathcal{E}}(\cdot \cdot)$	General conditional relative entropy by expectations in \mathcal{N}
$D_{\mathbb{M}}(\cdot \cdot)$	Measured relative entropy
$\hat{S}_{\text{BS}}(\cdot \cdot)$	BS-entropy
$S_f(\cdot \cdot)$	Standard f -divergence
$\hat{S}_f(\cdot \cdot)$	Maximal f -divergence

$EP(\cdot)$	Entropy production
$EP_A(\cdot)$	Conditional entropy production in A
$Cov(\cdot, \cdot)$	Covariance
$Cov_A(\cdot, \cdot)$	Conditional covariance in A
$Var(\cdot, \cdot)$	Variance
$Var_A(\cdot, \cdot)$	Conditional variance in A
$I_\rho(A : B)$	Mutual information of ρ
$I_\rho(A : B C)$	Conditional mutual information of ρ

Probability theory

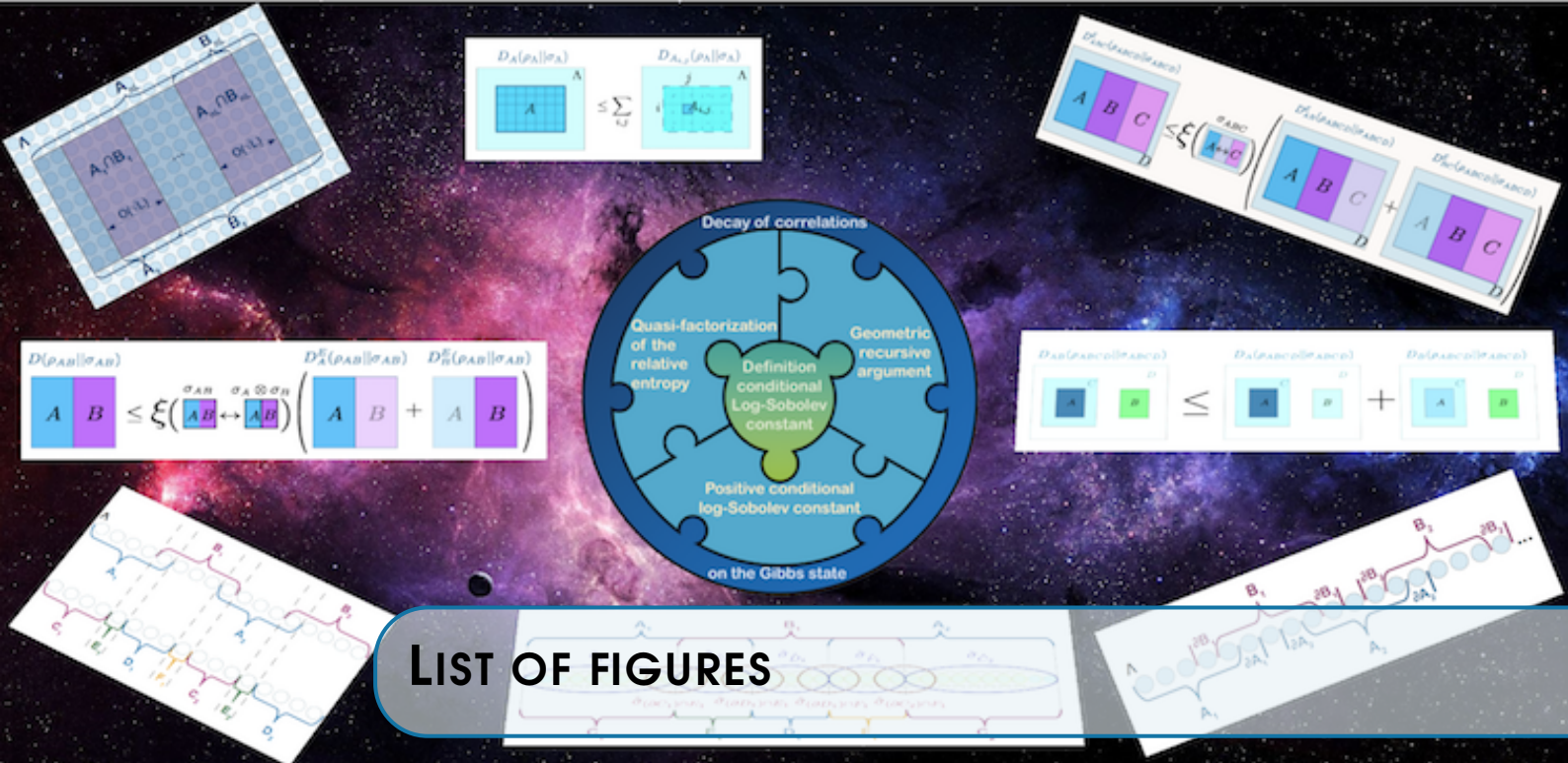
μ, ν	Probability measures
Ω	Configuration space
\mathcal{F}	σ -algebra
$(\Omega, \mathcal{F}, \mu)$	Probability space
$\mathcal{G} \subseteq \mathcal{F}$	Sub- σ -algebra
$Ent_\mu(\cdot)$	Entropy
$Ent_\mu(\cdot \mathcal{G})$	Conditional entropy
$H(\cdot \cdot)$	Relative entropy
$H_{\mathcal{G}}(\cdot \cdot)$	Conditional relative entropy
$X\Delta Y$	Symmetric difference
μ_Λ^τ	Gibbs measure
τ	Boundary condition

Miscellaneous

\mathbb{N}	Natural numbers
\mathbb{R}	Real numbers
\mathbb{C}	Complex numbers
\mathcal{R}_L	Set of rectangles of size L
Q_L	Cube of size L starting at the origin

Acronyms


QMS	Quantum Markov semigroup
QMC	Quantum Markov chain
CPTP	Completely positive and trace-preserving
MLSI	Modified logarithmic Sobolev inequality
DPI	Data processing inequality
RE	Relative entropy
CRE	Conditional relative entropy
CREexp	Conditional relative entropy by expectations
gCREexp	General conditional relative entropy by expectations
QF	Quasi-factorization
BS	Belavkin-Staszewski (for the BS-entropy)
RHS, LHS	Right-hand side, left-hand side



LIST OF FIGURES

1.1	Complete puzzle to prove the positivity of a logarithmic Sobolev constant	37
2.1	Puzzle completo para probar positividad de una constante logarítmica de Sobolev.	59
3.1	Splitting in A_n and B_n	85
4.1	Piece of the puzzle corresponding to the quasi-factorization of the relative entropy	109
4.2	Piece of the puzzle corresponding to the (weak) quasi-factorization of the relative entropy.	110
6.1	Identification between classical and quantum quantities when the states considered are classical.	134
7.1	Choice of indices in a tripartite Hilbert space $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$	138
7.2	Piece associated to the (weak) quasi-factorization of the relative entropy.	139
7.3	Graphical representation for the result of quasi-factorization obtained under the assumption of both ρ and σ tensor products.	140
7.4	Graphical representation for the result of quasi-factorization of the kind (QF-Ov) obtained under the assumption of σ_{ABC} tensor product.	141
7.5	Graphical representation for the result of quasi-factorization of the kind (QF-NonOv) obtained under the assumption of σ_{ABC} tensor product.	142
7.6	Graphical representation for the most general result of (weak) quasi-factorization under the assumption of σ_A tensor product.	143
7.7	Graphical representation for a result of quasi-factorization for the conditional relative entropy for arbitrary ρ_{ABC} and σ_{ABC}	145
7.8	Graphical representation for a result of quasi-factorization for the conditional relative entropy by expectations for arbitrary ρ_{AB} and σ_{AB}	147
8.1	Piece associated to the strong quasi-factorization of the relative entropy.	156
8.2	Graphical representation for the most general result of strong quasi-factorization under the assumption of σ_A tensor product.	157
8.3	System $ABCD$ where C shields A from BD	158

8.4	Graphical representation of the result of strong quasi-factorization under the assumption of σ_Λ a quantum Markov chain.	159
8.5	Clustering conditions to be assumed on the Gibbs state to prove positivity of the logarithmic Sobolev constant.	160
8.6	Piece associated to the strong quasi-factorization of the relative entropy.	164
8.7	Graphical representation of the result of quasi-factorization of Corollary 8.4.6. To simplify the notation, we have made here the following correspondence with the one of the corollary: $\Lambda \mapsto ABCD$, $A \mapsto AB$, $B \mapsto BC$	169
8.8	Complete puzzle to prove the positivity of a logarithmic Sobolev inequality . .	183
9.1	Piece for the definition of the conditional log-Sobolev constant	186
9.2	Pieces for the definitions of the conditional log-Sobolev constant and decay of correlations on the Gibbs state.	187
9.3	Pieces for the definitions of the conditional log-Sobolev constant and decay of correlations on the Gibbs state, and the result of quasi-factorization of the relative entropy. The piece of quasi-factorization appears with well-defined boundaries, as we showed in Section 7.2 that this result, indeed, can be seen as a strong quasi-factorization.	188
9.4	Pieces for the definitions of the conditional log-Sobolev constant and decay of correlations on the Gibbs state, and the results of quasi-factorization of the relative entropy and positivity of the conditional log-Sobolev constant.	190
9.5	Complete puzzle for the proof of positive log-Sobolev constants.	191
10.1	Piece for the definition of the conditional log-Sobolev constant	195
10.2	Splitting of Λ in fixed-sized subsets A_i and B_i , of which we just show the first four terms. We reduce for simplicity to the case $k = 2, l = 1$	199
10.3	Splitting of A in fixed-sized subsets A_i so that their boundaries do not overlap. For simplicity we restrict to the case $k = 2, l = 1$	201
10.4	Piece for the quasi-factorization of the relative entropy. Note that we use here the image that we usually devote to strong results of quasi-factorization. Although in the left-hand side of the inequality there is no conditional relative entropy, the result is stronger than we would directly obtain from Step 1, and this justifies the use of this image.	202
10.5	Piece for the geometric argument for the heat-bath dynamics.	203
10.6	Piece for the positivity of the conditional log-Sobolev constant for the heat-bath dynamics.	204
10.7	Complete puzzle for the positivity of the log-Sobolev constant for the heat-bath dynamics.	205
10.8	Conditions of decay of correlations on the Gibbs state.	206
10.9	Notation introduced in the splitting of Λ into size-fixed A_i and B_i for the discussion in Assumption 10.3.1. For simplicity we restrict to the case $k = 2, l = 1$	207
10.10	Decomposition of σ_Λ into the product of commuting terms for $k = 3$ and $l = 5$, assuming that Λ is decomposed only into A_1, B_1 and A_2 for simplification. . . .	208
11.1	Definition of the conditional log-Sobolev constant for the Davies dynamics. . .	217
11.2	Piece for the geometric recursive argument for the Davies dynamics.	218
11.3	Splitting in A and B	219
11.4	Splitting in A_n and B_n	220
11.5	Piece concerning the positivity of the conditional log-Sobolev constant.	224



SUMMARY

The fields of *quantum information theory* and *quantum many-body systems* have strong connections, as do their classical analogues. In the past years, we have come to see how results from each of the fields have generated an important impact on the other. On the one hand, tools developed in quantum information theory have helped to solve fundamental problems in condensed matter physics, whereas some new models created for many-body systems have been employed for the storage and transmission of quantum information. The connections between these fields and the interesting problems lying in their intersection are numerous.

One of the main goals in science nowadays, which is strongly connected to both of these fields, is the design and development of *quantum computers*. There is a huge scientific effort to build them and understand how to exploit their computational power to solve different problems. However, one of the main obstacles in the construction of large-scale quantum computers is the appearance of external *noise*, which should be controlled or suppressed. Some kinds of noise in quantum many body systems can be modelled by quantum dissipative evolutions which are governed by local Lindbladians. Their study is thus, fundamental for the field of theoretical and experimental quantum physics.

Another big obstacle in the construction of a quantum computer is the design of *lifetime quantum memories*. In the theoretical proposal of dissipative state engineering, made in 2009, by Verstraete et al. [VWC09] and Kraus et al. [Kra+13], they proposed the idea that a robust way of constructing interesting quantum systems which preserve the coherence for longer periods might be based on the same quantum dissipative systems. They base this proposal precisely in the dissipative nature of noise, since it eliminates the problem of having to initialize the system carefully, due to the fact that the system is driven to a stationary fixed state that is independent of the initial state. Moreover, some experimental results of the past few years have given value to this proposal, inducing a remarkable growth in the interest on such systems.

Therefore, one of the main problems nowadays lying in the intersection between the fields of quantum information theory and quantum many-body systems is the problem of *thermalization*, i.e., the study of how a thermal quantum dissipative evolution converges to its thermal equilibrium. It has recently generated great interest in both communities for several reasons, one of them being the uprising number of tools available from quantum information theory [RGE12] [Mül+15] to

The image above shows a beautiful sunset behind the amazing view of the city of Taipei from the Elephant Mountain, which I could witness during my research stay of 2 months and a half in *Academia Sinica* in the summer of 2018.

address two important problems concerning thermalization: The study of the conditions under which a system thermalizes in the infinite limit, and how fast this thermalization occurs.

In this thesis, we focus on the latter, namely how fast a dissipative system thermalizes. This “velocity” of thermalization can be studied by means of the *mixing time*, i.e., the time that it takes for every initial state undergoing a dissipative evolution to become almost indistinguishable from the thermal equilibrium state. In particular, we are interested in physical systems for which this convergence is fast enough, in a regime that is called *rapid mixing*. The problem of finding bounds for the mixing time, and thus, conditions for rapid mixing to hold, can be addressed via the optimal constants associated to some quantum functional inequalities, such as the *spectral gap* (for the Poincaré inequality) [Tem+10] or the *log-Sobolev constant* (for the log-Sobolev inequality) [KT16]. Here, we focus on the latter.

The main aim of this thesis is to provide **sufficient conditions on the fixed point of a quantum dissipative evolution so that the system has a positive log-Sobolev constant**. This problem was previously addressed in the classical setting. In [DPP02], it was shown that a classical spin system in a lattice, for a certain dynamics and under some clustering conditions in the Gibbs measure associated to this dynamics, has a positive log-Sobolev constant. This result notably simplified the previous work in [MO94a] via a result of *quasi-factorization of the relative entropy* in terms of a *conditional entropy*. Previously, a result of *quasi-factorization of the variance* [BCC02] had been used to prove positivity of the spectral gap for certain dynamics, under some conditions in the Gibbs measure.

The latter found its quantum analogue in [KB16], where the notion of *conditional spectral gap* was introduced and the positivity of the spectral gap for the Davies and heat-bath dynamics associated to a local commuting Hamiltonian was proven, via a result of quasi-factorization of the variance, under a condition of strong clustering of correlations on the Gibbs state. In this thesis, our purpose is to study the quantum analogue of the classical proof of positivity for log-Sobolev constants in classical spin systems via results of quasi-factorization of the entropy, obtaining thus an exponential improvement in the dependence with the system size with respect to the spectral gap case.

Moreover, since positivity of the log-Sobolev constant implies positivity of the spectral gap for a certain dynamics [KT16], we focus on the heat-bath and Davies generators, for which the spectral gap has already been studied in the commuting case. These generators constitute classes of Gibbs samplers in the setting of quantum systems, which are used to develop simulation and sampling algorithms that can be used to prepare large classes of thermal states of physically relevant Hamiltonians. More specifically, the Davies generator is derived from the weak coupling of a finite quantum system to a large thermal bath, whereas the heat-bath generator is constructed following the same idea than for the classical heat-bath Monte-Carlo algorithm.

For these dynamics, in this text we address the following two main objectives:

1. **Develop a strategy to prove that a quantum system has a positive log-Sobolev constant, via results of quasi-factorization of the relative entropy.**
2. **Apply that strategy for the heat-bath and the Davies dynamics, to obtain positivity of log-Sobolev constants, under some conditions on the fixed points of the evolutions.**

For the first point, building on results for classical spin systems, we develop a strategy of five steps to prove that a quantum dissipative system has a positive log-Sobolev constant, which implies a tight bound on its mixing time. For the second point, after introducing and characterizing in several ways the notion of conditional relative entropy, we prove different results of quasi-factorization of the relative entropy, which we subsequently employ to prove positivity for the log-Sobolev constant for the heat-bath and Davies dynamics, under some conditions of clustering of correlations on the fixed points of the evolutions.



RESUMEN

Los campos de *teoría de la información cuántica* y *sistemas cuánticos de muchos cuerpos* tienen fuertes conexiones, al igual que sus análogos clásicos. En los últimos años, hemos llegado a ver cómo resultados de cada uno de los campos han generado un gran impacto en el otro. Por una parte, algunas herramientas desarrolladas en teoría de la información cuántica han ayudado para resolver problemas fundamentales en física de la materia condensada, mientras que nuevos modelos creados para sistemas de muchos cuerpos se han empleado para el almacenamiento y transmisión de información cuántica. Las conexiones entre estos campos y los interesantes problemas que se encuentran en su intersección son numerosos.

Una de las grandes metas en ciencia actualmente, que está fuertemente relacionada con ambos campos, es el diseño y desarrollo de un *ordenador cuántico*. Se está haciendo un gran esfuerzo científico para construir estos ordenadores y entender cómo explotar su potencia computacional para resolver distintos problemas. Sin embargo, uno de los principales obstáculos en la construcción de ordenadores cuánticos a gran escala es la aparición de *ruido* externo, que debería ser controlado o suprimido. Algunos tipos de ruido en sistemas cuánticos de muchos cuerpos se pueden modelar con evoluciones disipativas cuánticas que están gobernadas por Lindbladianos locales. Su estudio es, por tanto, fundamental para los campos de física cuántica teórica y experimental.

Otro gran obstáculo en la construcción de un ordenador cuántico es el diseño de *memorias cuánticas duraderas*. En la propuesta teórica de ingeniería disipativa de estados, hecha en 2009 por Verstraete et al. [VWC09] y Kraus et al. [Kra+13], se propuso la idea de que una forma robusta de construir sistemas cuánticos interesantes que preserven la coherencia durante periodos más largos podría estar basada en los mismos sistemas disipativos cuánticos. Esta propuesta se basa precisamente en la naturaleza disipativa del ruido, puesto que elimina el problema de tener que inicializar el sistema cuidadosamente, debido a que el sistema converge hacia un estado estacionario fijo independientemente del estado inicial. Además, algunos resultados experimentales de los últimos años han reforzado esta propuesta, dando lugar a un gran crecimiento en el interés en estos sistemas.

Por tanto, uno de los principales problemas actuales en la intersección entre los campos de teoría de la información cuántica y los sistemas cuánticos de muchos cuerpos es el problema de la *termalización*, es decir, el estudio de cómo una evolución disipativa cuántica termal converge a su estado térmico de equilibrio. Recientemente ha generado gran interés en ambas comunidades

Same landscape than in the previous image, now after the sunset.

por varias razones, siendo una de ellas el aumento en el número de herramientas disponibles de teoría de la información cuántica [RGE12] [Mül+15] para afrontar dos importantes problemas relativos a la termalización: El estudio de condiciones bajo las cuales un sistema termaliza en el límite infinito, y cómo de rápido se produce esta termalización.

En esta tesis, nos centramos en el último punto, es decir, en cómo de rápido termaliza un sistema disipativo. Esta “velocidad” de termalización se puede estudiar a partir del *tiempo de equilibración*, es decir, el tiempo que tarda cada estado inicial que sufre una evolución disipativa en convertirse casi indistinguible del estado de equilibrio térmico. En particular, estamos interesados en sistemas físicos para los que la convergencia es suficientemente rápida, en un régimen que llamamos *equilibración rápida*. El problema de encontrar cotas para el tiempo de equilibración, y, por tanto, condiciones para que haya equilibración rápida, se puede afrontar desde el punto de vista de constantes óptimas asociadas a algunas desigualdades funcionales cuánticas, como el *gap espectral* (para la desigualdad de Poincaré) [Tem+10] o la *constante de log-Sobolev* (para la desigualdad de log-Sobolev) [KT16]. Aquí nos centramos en esta última.

El principal objetivo de esta tesis es proporcionar **condiciones suficientes en el punto fijo de una evolución disipativa cuántica para que el sistema tenga una constante de log-Sobolev positiva**. Este problema se estudió previamente en el caso clásico. En [DPP02], se mostró que un sistema de espines clásico en una retícula, para una cierta dinámica y bajos ciertas condiciones de agrupamiento en la medida de Gibbs asociada a esta dinámica, tiene una constante de log-Sobolev positiva. Este resultado simplificó notablemente el trabajo previo de [MO94a] a partir de un resultado de *quasi-factorización de la entropía relativa* en función de una *entropía condicionada*. Previamente, un resultado de *quasi-factorización de la varianza* [BCC02] se había usado para probar la positividad del gap espectral para ciertas dinámicas, bajo algunas condiciones en la medida de Gibbs.

Este último resultado encontró su análogo cuántico en [KB16], donde se introdujo la noción de *gap espectral condicionado* y se probó la positividad del gap espectral para las dinámicas de Davies y *heat-bath* asociadas a un Hamiltoniano conmutante local, a partir de un resultado de quasi-factorización de la varianza, bajo ciertas condiciones fuertes de agrupamiento de correlaciones en el estado de Gibbs. En esta tesis, nuestro objetivo es estudiar el análogo cuántico de la prueba clásica de positividad para constantes de log-Sobolev en sistemas de espines clásicos vía resultados de quasi-factorización de la entropía, obteniendo en consecuencia una mejora exponencial en la dependencia con el tamaño del sistema con respecto al caso del gap espectral.

Además, puesto que la positividad en la constante de log-Sobolev implica positividad en el gap espectral para una cierta dinámica [KT16], nos centramos en las dinámicas de *heat-bath* y Davies, para las que el gap espectral ya se ha estudiado en el caso conmutante. Estos generadores constituyen clases de muestreadores de Gibbs en el campo de sistemas cuánticos, que se emplean para desarrollar algoritmos de simulación y muestreo, los cuales se pueden utilizar para preparar grandes clases de estados térmicos de Hamiltonianos físicamente relevantes. Más específicamente, el generador de Davies se deriva del acoplamiento débil de un sistema cuántico finito con un baño térmico grande, mientras que el generador de *heat-bath* se construye siguiendo la misma idea que para el algoritmo clásico de Monte-Carlo de *heat-bath*.

Para estas dinámicas, nos planteamos los dos siguientes objetivos principales en este texto:

1. **Desarrollar una estrategia para probar que un sistema cuántico tiene una constante de log-Sobolev positiva a partir de resultados de quasi-factorización de la entropía relativa.**
2. **Aplicar dicha estrategia para las dinámicas de *heat-bath* y Davies, obteniendo positividad en las constantes de log-Sobolev bajo ciertas condiciones en los puntos fijos de las evoluciones.**

Para el primer punto, partiendo de resultados para sistemas de espines clásicos, desarrollamos una estrategia de cinco pasos para probar que un sistema disipativo cuántico tiene una constante de log-Sobolev positiva, lo cual implica una cota fina en su tiempo de equilibración. Para el segundo punto, tras introducir y caracterizar de varias formas la noción de entropía relativa condicionada, probamos diferentes resultados de quasi-factorización de la entropía relativa, los cuales posteriormente empleamos para probar positividad de las constantes de log-Sobolev para las dinámicas, bajo ciertas condiciones de agrupamiento de correlaciones en los puntos fijos de las evoluciones.

Part I

INTRODUCTION AND PRELIMINARIES



1. INTRODUCTION

1.1 NOTATION AND BACKGROUND

1.1.1 NOTATION

Let us fix some notations that we will use throughout this manuscript, although some objects will be presented in more detail later on.

In this text, we consider finite-dimensional complex Hilbert spaces. For Λ a set of $|\Lambda|$ parties, we denote the multipartite finite-dimensional Hilbert space of $|\Lambda|$ parties by

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x,$$

where \mathcal{H}_x is a finite-dimensional Hilbert space associated to each site x of the lattice. We will denote by $|\psi\rangle$ a vector in \mathcal{H}_x and by $\langle\psi|$ its adjoint.

Throughout this text, Λ will often consist of 3 parties, and we will denote by $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ the corresponding tripartite Hilbert space. Furthermore, a substantial part of the thesis concerns quantum spin lattice systems and we often assume that $\Lambda \subset \mathbb{Z}^d$ is a finite subset. In general, we use uppercase Latin letters to denote systems or sets.

For every finite-dimensional \mathcal{H}_Λ , we denote the associated set of bounded linear operators by $\mathcal{B}_\Lambda := \mathcal{B}(\mathcal{H}_\Lambda)$, and by $\mathcal{A}_\Lambda := \mathcal{A}(\mathcal{H}_\Lambda)$ its subset of *observables*, i.e. Hermitian operators, which we denote by lowercase Latin letters. We further denote by

$$\mathcal{S}_\Lambda := \mathcal{S}(\mathcal{H}_\Lambda) = \{f_\Lambda \in \mathcal{A}_\Lambda : f_\Lambda \geq 0 \text{ and } \text{tr}[f_\Lambda] = 1\}$$

the set of *density matrices*, or *states*, and denote its elements by lowercase Greek letters. In particular, whenever they appear in the text, *Gibbs states* are denoted by σ_Λ . We usually denote the space where each operator is defined using the same subindex as for the space, but we might drop it when it is unnecessary.

We write $\mathbb{1}$ for the identity matrix and id for the identity operator. For bipartite spaces $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, we consider the natural inclusion $\mathcal{A}_A \hookrightarrow \mathcal{A}_{AB}$ by identifying each operator $f_A \in \mathcal{A}_A$ with $f_A \otimes \mathbb{1}_B$.

Beautiful landscape of Seefeld in Tirol (Austria) during the *3rd Seefeld workshop on Quantum Information*, in July 2016. It is the first workshop that I attended outside of Spain, and that is the reason to be the perfect image for the “introduction” to the thesis.

Given a bipartite Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, we define the *partial trace over A* as the unique linear map $\text{tr}_A : \mathcal{B}_{AB} \rightarrow \mathcal{B}_B$ such that $\text{tr}_A[a \otimes b] = b \text{tr}[a]$ for all $a \in \mathcal{B}_A$ and $b \in \mathcal{B}_B$. Moreover, we define the modified partial trace in A of $f_{AB} \in \mathcal{A}_{AB}$ by $\text{tr}_A[f_{AB}] \otimes \mathbb{1}_B$, but we denote it by $\text{tr}_A[f_{AB}]$ in a slight abuse of notation. Moreover, we say that an operator $g_{AB} \in \mathcal{A}_{AB}$ has *support* in A if it can be written as $g_A \otimes \mathbb{1}_B$ for some operator $g_A \in \mathcal{A}_A$. Note that given $f_{AB} \in \mathcal{A}_{AB}$, we write $f_A := \text{tr}_B[f_{AB}]$.

A *quantum channel* [Wol12] is a completely positive and trace-preserving map. We call a linear map $\mathcal{T} : \mathcal{B}_\Lambda \rightarrow \mathcal{B}_\Lambda$ a *superoperator*, and say that it is *positive* if it maps positive operators to positive operators. Moreover, we call \mathcal{T} *completely positive* if, given \mathcal{M}_n the space of complex $n \times n$ matrices, $\mathcal{T} \otimes \text{id} : \mathcal{B}_\Lambda \otimes \mathcal{M}_n \rightarrow \mathcal{B}_\Lambda \otimes \mathcal{M}_n$ is positive for every $n \in \mathbb{N}$. Finally, we say that \mathcal{T} is *trace preserving* if $\text{tr}[\mathcal{T}(f_\Lambda)] = \text{tr}[f_\Lambda]$ for all $f_\Lambda \in \mathcal{B}_\Lambda$.

For \mathcal{M} and \mathcal{N} two von Neumann algebras, if we consider an operator $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$, we denote by $\mathcal{T}^* : \mathcal{N} \rightarrow \mathcal{M}$ its dual map. Furthermore, we will frequently consider \mathcal{B}_Λ equipped with the *Hilbert-Schmidt inner product*, i.e.,

$$\langle A, B \rangle_{\text{HS}} := \text{tr}[A^* B] \quad \text{for } A, B \in \mathcal{B}_\Lambda,$$

where A^* represents the adjoint of A . Sometimes we will also consider a σ -*weighted inner product*, given by

$$\langle A, B \rangle_\sigma := \text{tr}[\sigma^{1/2} A^* \sigma^{1/2} B] \quad \text{for } A, B \in \mathcal{B}_\Lambda.$$

In general, we will denote by $\|\cdot\|_{\mathbb{L}_p(\sigma)}$ the σ -*weighted \mathbb{L}_p norms*, and by $\|\cdot\|_p$ the *Schatten p -norms*, given by $\text{tr}[|\cdot|^p]^{1/p}$. In particular, we denote by $\|\cdot\|_\infty$ the usual operator norm, as well as by $\|\cdot\|_1 = \text{tr}[|\cdot|]$ the trace-norm.

Given $x, y \in \Lambda \subset \mathbb{Z}^d$, we denote by $d(x, y)$ the Euclidean distance between x and y in \mathbb{Z}^d . Hence, the distance between two subsets of Λ , A and B , is given by

$$d(A, B) := \min\{d(x, y) : x \in A, y \in B\}.$$

Finally, let us comment on the fact that we restrict to finite-dimensional Hilbert spaces. In many quantum mechanical systems, the observables are operators on an infinite-dimensional, but separable, Hilbert space \mathcal{H} . One can extend easily the definition of the trace, and hence of density matrices, to this infinite-dimensional setting. However, it is relatively easy to show that any density operator on \mathcal{H} (positive semidefinite operator with trace 1) is a compact one, and hence, since the approximation property is satisfied by this class of spaces, it can be approximated by finite-rank operators in the operator norm. Although this does not imply that results in finite-dimensional Hilbert spaces are extended to infinite-dimensional ones in a straightforward way, at least this discussion provides an argument to notice that the essential aspects of the inequalities for density matrices that will appear throughout this text are contained in the finite-dimensional case (see [Car09]).

1.1.2 QUANTUM DISSIPATIVE EVOLUTIONS

The postulates of quantum mechanics state that an isolated physical systems is completely described by a density operator on a complex Hilbert space which is known as the state space of the system. Moreover, the evolution of the isolated system is described by unitary transformations, i.e., if the physical properties of the system are encoded in the density operator ρ , then the evolution of the system is given by $U\rho U^*$, with U a unitary operator. This evolution is clearly reversible, and its inverse is given by $U^*\rho U$.

However, this theoretical class of systems does not constitute a suitable approach for modeling and studying realistic situations for microscopic systems, since any real quantum many-body

system is contained in a huge thermal bath, which is extremely complicated to model and with which there exist unavoidable interactions. Indeed, no real experiment can be executed at zero temperature or be completely shielded from noise. Then, for a more realistic approach, we need to focus on open quantum systems, i.e., many-body systems which are surrounded by an environment with which there exist unavoidable interactions. The resulting dynamics is then dissipative and, in particular, irreversible.

Let us describe the evolution of an open quantum many-body system. Assume that \mathcal{H} is the finite-dimensional Hilbert space associated to a certain quantum system and consider a map $\Phi: \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H})$ to describe its evolution. We can justify that Φ has to be a quantum channel in two different ways, taking into account the properties that a physically realizable evolution should satisfy.

First, Φ should satisfy the following properties:

- It should map states to states, which implies that it should be linear, positive and trace preserving.
- If we consider $\rho \in \mathcal{S}(\mathcal{H})$ and $\sigma \in \mathcal{S}(\mathcal{H}')$ in a different Hilbert space with trivial evolution, the composite map

$$\begin{aligned} \hat{\Phi}: \mathcal{S}(\mathcal{H} \otimes \mathcal{H}') &\rightarrow \mathcal{S}(\mathcal{H} \otimes \mathcal{H}') \\ \rho \otimes \sigma &\mapsto \Phi(\rho) \otimes \sigma, \end{aligned}$$

where $\hat{\Phi} = \Phi \otimes \text{id}$ should be positive.

Since this should hold for \mathcal{H}' of any dimension, $\hat{\Phi}$ is completely positive.

Putting all these properties together, we conclude that $\hat{\Phi}$ is a quantum channel.

Another way to justify this fact is the following. Although we are focusing now on open quantum many-body systems, which consist on systems interacting with an environment, the pair system-environment does constitute a closed system. Therefore, both of them together evolve by means of a unitary operator as mentioned above.

More specifically, if we assume that the state associated to the environment is a pure one, $|\psi\rangle\langle\psi|_E$, then the evolution of an initial state ρ in the system is given by

$$\rho \mapsto \rho \otimes |\psi\rangle\langle\psi|_E \mapsto U(\rho \otimes |\psi\rangle\langle\psi|_E)U^* \mapsto \text{tr}_E[U(\rho \otimes |\psi\rangle\langle\psi|_E)U^*] \equiv \tilde{\rho},$$

where we trace out the environment, tr_E , to obtain the effect on the system. It is easy to notice that each one of the steps appearing above constitutes a quantum channel, and thus the evolution of the open system is described by a quantum channel. More information on the fact that every quantum channel can be seen as a restricted action of an evolution by means of a unitary will appear on Chapter 12, where we will use this fact to construct results for quantum channels from conditional expectations using Stinespring's dilation theorem.

This discussion is valid for every step of the dynamical evolution. For the continuous-time description, note that for every instant $t \geq 0$, the corresponding time slice is a realizable evolution \mathcal{T}_t and thus a quantum channel.

To construct the continuous-time evolution from this, we assume that the effect of the environment on the system is almost irrelevant, but has to be taken into account. However, due to the small effect on the system, we can assume that the environment does not evolve with time, which is a reasonable assumption, for instance, if the environment is a heat-bath which is much larger than the system and whose temperature is going to be practically invariant.

Then, we can assume the *weak-coupling limit*, which means that in the evolution of the pair system-environment mentioned above we can consider that $U(\rho \otimes |\psi\rangle\langle\psi|_E)U^*$ is very close to $\tilde{\rho} \otimes |\psi\rangle\langle\psi|_E$ for a certain $\tilde{\rho}$.

Moreover, we further reduce our study to the case in which the environment holds no memory and thus the future evolution only depends on the present. This is called the *Markovian approximation*.

Considering all these assumptions and discussions, we can define a *quantum dissipative evolution* (or *quantum Markov semigroup*) as a 1-parameter continuous semigroup $\{\mathcal{T}_t^*\}_{t \geq 0}$ of completely positive, trace-preserving maps, verifying:

- $\mathcal{T}_0^* = \mathbb{1}$.
- $\mathcal{T}_t^* \circ \mathcal{T}_s^* = \mathcal{T}_{t+s}^*$,

for every $t, s \geq 0$, where we are using the notation $*$ to emphasize the fact that we are considering the Schrödinger picture.

Given a finite lattice $\Lambda \subset \mathbb{Z}^d$, the generator of this semigroup is denoted by \mathcal{L}_Λ^* and called *Lindbladian* (or *Liouvillian*), since its dual version in the Heisenberg picture (for observables) satisfies the Lindblad (or GKLS) form [Lin76], [GKS76] for every $X_\Lambda \in \mathcal{B}_\Lambda$:

$$\mathcal{L}_\Lambda(X_\Lambda) = i[H, X_\Lambda] + \frac{1}{2} \sum_{k=1}^l [2L_k^* X_\Lambda L_k - (L_k^* L_k X_\Lambda + X_\Lambda L_k^* L_k)],$$

where $H \in \mathcal{A}_\Lambda$, the $L_k \in \mathcal{B}_\Lambda$ are the *Lindblad operators* and $[\cdot, \cdot]$ denotes the commutator. Moreover, it is called Liouvillian for satisfying Liouville's equation, i.e.:

$$\frac{d}{dt} \mathcal{T}_t^* = \mathcal{L}_\Lambda^* \circ \mathcal{T}_t^* = \mathcal{T}_t^* \circ \mathcal{L}_\Lambda^*.$$

Thus, we can write the elements of the quantum Markov semigroup as

$$\mathcal{T}_t^* = e^{t\mathcal{L}_\Lambda^*}.$$

1.1.3 MIXING TIME AND LOG-SOBOLEV INEQUALITIES

Consider again the lattice Λ mentioned above. Given $\rho_\Lambda \in \mathcal{S}_\Lambda$, let us denote

$$\rho_t := \mathcal{T}_t^*(\rho_\Lambda)$$

for every $t \geq 0$ (when the omission of the subindex Λ does not cause any confusion). With this notation, it is clear that the evolution of the system can be rewritten as the quantum dynamical master equation:

$$\partial_t \rho_t = \mathcal{L}_\Lambda^*(\rho_t).$$

We say that a certain state σ_Λ is an *invariant state* of $\{\mathcal{T}_t^*\}_{t \geq 0}$ if

$$\sigma_t := \mathcal{T}_t^*(\sigma_\Lambda) = \sigma_\Lambda$$

for every $t \geq 0$.

Throughout all this thesis, we will restrict to the *primitive* case, i.e., $\{\mathcal{T}_t^*\}_{t \geq 0}$ has a unique full-rank invariant state (and thus there is a unique σ_Λ for which $\mathcal{L}_\Lambda^*(\sigma_\Lambda) = 0$). Let us further assume that the quantum Markov process studied is *reversible*, i.e., satisfies the *detailed balance condition*

$$\langle f, \mathcal{L}_\Lambda(g) \rangle_{\sigma_\Lambda} = \langle \mathcal{L}_\Lambda(f), g \rangle_{\sigma_\Lambda}$$

for every $f, g \in \mathcal{A}_\Lambda$, where \mathcal{L}_Λ is the generator of the evolution semigroup in the Heisenberg picture. An interesting problem concerning quantum Markov semigroups is the study of the speed of convergence to this unique invariant state, which can be done by bounding the mixing time.

The *mixing time* of a quantum Markov semigroup is the time that it takes for an initial state to become almost indistinguishable from the invariant state, i.e., the fixed point of the evolution. More specifically, for every $\varepsilon > 0$, it is given by the following expression

$$\tau(\varepsilon) := \min \left\{ t > 0 : \sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \|\rho_t - \sigma_\Lambda\|_1 \leq \varepsilon \right\}. \quad (1.1)$$

Moreover, there is a special class of generators for which this convergence is “fast enough”, which we call rapid mixing. In words, we say that \mathcal{L}_Λ^* satisfies *rapid mixing* if

$$\sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \|\rho_t - \sigma_\Lambda\|_1 \leq \text{poly}(|\Lambda|) e^{-\gamma t},$$

for a constant $\gamma > 0$ and where $\text{poly}(|\Lambda|)$ stands for a polynomial in the size of Λ . This property has profound implications in the system, such as stability against external perturbations [Cub+15] and the fact that its fixed points satisfy an area law for the mutual information [Bra+15a].

As mentioned above, a fundamental problem lying in the interesection between quantum information theory and condensed matter physics is the speed of convergence of this kind of dissipative evolutions to their equilibrium states, or fixed points, which can be done by studying the mixing time of the evolution. Different bounds for the mixing time can be obtained by means of the optimal constants for some quantum functional inequalities, such as the *spectral gap* for the *Poincaré inequality* [Tem+10] and the *logarithmic Sobolev constant* for the *logarithmic Sobolev inequality* [KT16]. In this thesis we will focus on the latter.

There exists a whole family of *logarithmic Sobolev inequalities* (*log-Sobolev inequalities* for short), which can be indexed by an integer parameter, as done in [KT16]. This text concerns the so-called 1-log-Sobolev inequality, also known in the literature as modified log-Sobolev inequality (MLSI) and which we will call throughout the text just simply *log-Sobolev inequality* (with associated *log-Sobolev constant*), since there is no possible confusion. Note, however, that this notion does not correspond classically to the classical logarithmic Sobolev constant, although the quantum 2-log-Sobolev constant does.

In detail, given $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \rightarrow \mathcal{S}_\Lambda$ a primitive, reversible Lindbladian with fixed point $\sigma_\Lambda \in \mathcal{S}_\Lambda$, we define the *log-Sobolev constant* of \mathcal{L}_Λ^* by

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\text{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)},$$

where $D(\rho_\Lambda || \sigma_\Lambda)$ is the *relative entropy* between ρ_Λ and σ_Λ and is given by

$$D(\rho_\Lambda || \sigma_\Lambda) := \text{tr}[\rho_\Lambda(\log \rho_\Lambda - \log \sigma_\Lambda)].$$

The log-Sobolev constant, if positive, provides an upper bound for the mixing time of a dissipative evolution. The derivation of the bound obtained for the mixing time in terms of log-Sobolev constants can be explicitly found in Section 4.5, and yields the expression:

$$\|\rho_t - \sigma_\Lambda\|_1 \leq \sqrt{2 \log(1/\sigma_{\min})} e^{-\alpha(\mathcal{L}_\Lambda^*) t}. \quad (1.2)$$

This bound provides an exponential improvement with respect to a bound in terms of the spectral gap (see the discussion about this topic in [KT16]). Moreover, in the same paper the authors showed that the former implies the latter, i.e., if a system has a positive log-Sobolev constant, then it also has a positive spectral gap. Therefore, it is reasonable to tackle the problem of finding systems which have a positive log-Sobolev constant amongst the class of systems for which a positive spectral gap has already been proven to exist.

Proving whether a Lindbladian has a positive log-Sobolev constant is, thus, a fundamental problem in open quantum many-body systems. In this line, the main aim of this thesis is:

Find sufficient static conditions on the fixed point of a dissipative evolution so that the system has a positive log-Sobolev constant.

In turn, this provides conditions under which a system satisfies rapid mixing, a property with numerous implications in quantum information theory, as previously discussed.

This problem was previously addressed in the classical setting. As we will show below, in [DPP02], the authors showed that a classical spin system in a lattice, for a certain dynamics and a clustering condition in the Gibbs measure associated to this dynamics, has a positive log-Sobolev constant. This result, inspired by the seminal work of Martinelli and Olivieri [MO94a], aimed to simplify notably their proof via a result of quasi-factorization of the entropy. Previously, a result of quasi-factorization of the variance [BCC02] had been used to prove positivity of the spectral gap for certain dynamics.

The latter found its quantum analogue in [KB16], where the authors proved positivity of the spectral gap for the Davies and heat-bath dynamics associated to a local commuting Hamiltonian, via a result of quasi-factorization of the variance, under a condition of strong clustering in the Gibbs state. These generators constitute classes of Gibbs samplers in the setting of quantum systems, which are used to develop simulation and sampling algorithms that can be used to prepare large classes of thermal states of physically relevant Hamiltonians. More specifically, the Davies generator is derived from the weak coupling of a finite quantum system to a large thermal bath, whereas the heat-bath generator is constructed following the same idea than for the classical heat-bath Monte-Carlo algorithm.

Because of the positive results obtained for the spectral gap for these two classes of dynamics, we will only address in this text the problem of proving positivity of log-Sobolev constants for the heat-bath dynamics and the Davies dynamics.

Following the aim introduced above, we can state the two main objectives of this thesis as follows:

1. **Develop a strategy to prove that a quantum system has a positive log-Sobolev constant, via results of quasi-factorization of the relative entropy.**
2. **Test that strategy for the heat-bath and the Davies dynamics, under some conditions on the fixed points of the evolutions.**

In the next section, we will address the first of these objectives. More specifically, building from results for classical spin systems, we will conceive and implement a strategy to prove that a quantum system has a positive log-Sobolev constant based on several steps, some of them concerning suitable definitions for certain concepts, and some others consisting of some results that need to be proven.

Subsequently, in Section 1.3, we will comment on the main results obtained in this thesis in the line of the second objective introduced above, i.e., results aimed at proving positivity of log-Sobolev constants for heat-bath or Davies dynamics based on the strategy previously introduced.

1.2 STRATEGY TO FIND POSITIVE LOG-SOBOLEV CONSTANTS

The problem of proving whether a certain system has a positive log-Sobolev constant was previously addressed for classical spin systems. In [DPP02], the authors showed that a classical spin system in a lattice, for a certain dynamics and a clustering condition in the Gibbs measure associated to this dynamics, satisfies a *modified logarithmic Sobolev inequality*, or *entropy inequality*, whose quantum analogue we call in this text just *log-Sobolev inequality*.

In [Ces01], the usual *logarithmic Sobolev inequality*, corresponding to the *2-log-Sobolev inequality* in the quantum case, was studied via another similar condition of clustering in the

Gibbs measure. Both results were inspired by the seminal work of Martinelli and Olivieri [Mar99], [MO94a], [MO94b] and aimed to simplify notably their proof via a result of quasi-factorization of the entropy in terms of some conditional entropies. Previously, a result of quasi-factorization of the variance [BCC02] had been used to prove positivity of the spectral gap for certain dynamics, under certain conditions in the Gibbs measure.

Let us focus now on the main result of [DPP02] mentioned above, and more specifically in the strategy followed there. This result will be discussed in detail in Chapter 3. For the time being, let us just introduce some basic notions and briefly explain the different steps followed in the proof.

Consider a probability space $(\Omega, \mathcal{F}, \mu)$. For every f in Ω with $f > 0$, the *entropy* of f is defined by

$$\text{Ent}_\mu(f) := \mu(f \log f) - \mu(f) \log \mu(f).$$

Moreover, considering \mathcal{L}_Λ^τ to be the Markov generator of the stochastic dynamics studied in [DPP02], for $\Lambda \subset \subset \mathbb{Z}^d$ a finite lattice and $\tau \in \Omega$ a boundary condition, the *Dirichlet form* associated to \mathcal{L}_Λ^τ is given by

$$\mathcal{E}_\Lambda^\tau(f, g) := -\mu_\Lambda^\tau(f \mathcal{L}_\Lambda^\tau g),$$

where μ_Λ^τ is the *Gibbs measure* in Λ with boundary condition τ , which corresponds to the unique invariant measure for the dynamics, and whose quantum analogue in our results will be a Gibbs state.

Then, we can define the *log-Sobolev constant* (which appears as *entropy constant* in [DPP02]) by

$$\alpha(\mathcal{L}_\Lambda^\tau) := \inf \left\{ \frac{\mathcal{E}_\Lambda^\tau(f, \log f)}{\text{Ent}_{\mu_\Lambda^\tau}(f)} : f \geq 0, f \log f \in \mathcal{L}_1(\mu_\Lambda^\tau), \text{Ent}_{\mu_\Lambda^\tau}(f) \neq 0 \right\}. \quad (\text{Log-Sob})$$

Now, going back to a more general probability space $(\Omega, \mathcal{F}, \mu)$ and given a sub- σ -algebra $\mathcal{G} \subseteq \mathcal{F}$, a *conditional entropy* in \mathcal{G} is defined as in the following way for every $f > 0$:

$$\text{Ent}_\mu(f | \mathcal{G}) := \mu(f \log f | \mathcal{G}) - \mu(f | \mathcal{G}) \log \mu(f | \mathcal{G}),$$

where $\mu(f | \mathcal{G})$ is given by

$$\int_G \mu(f | \mathcal{G}) d\mu = \int_G f d\mu \quad \text{for each } G \in \mathcal{G}.$$

With these definitions, in [DPP02], they first prove the a result of quasi-factorization of the entropy. Indeed, given $\mathcal{F}_1, \mathcal{F}_2$ sub- σ -algebras of \mathcal{F} , and assuming that there exists a probability measure $\bar{\mu}$ that makes \mathcal{F}_1 and \mathcal{F}_2 independent, $\mu \ll \bar{\mu}$ and $\mu | \mathcal{F}_i = \bar{\mu} | \mathcal{F}_i$ for $i = 1, 2$, they prove for every $f \geq 0$ such that $f \log f \in L^1(\mu)$ and $\mu(f) = 1$ that the following inequality holds:

$$\text{Ent}_\mu(f) \leq \frac{1}{1 - 4\|h - 1\|_\infty} \mu [\text{Ent}_\mu(f | \mathcal{F}_1) + \text{Ent}_\mu(f | \mathcal{F}_2)], \quad (\text{QF})$$

where $h = \frac{d\mu}{d\bar{\mu}}$ is the Radon-Nikodym derivative of μ with respect to $\bar{\mu}$, and thus measures in some sense how far is μ from making \mathcal{F}_1 and \mathcal{F}_2 independent.

Subsequently, given an initial lattice $\Lambda \subset \subset \mathbb{Z}^d$, they devise a certain geometric splitting for Λ in terms of some overlapping subregions which allows to reduce the log-Sobolev constant in Λ to the log-Sobolev constant of a subregion with small size. More esepcifically, given a family

of d -dimensional rectangular subregions of Λ whose largest side has size L and whose smallest side is not smaller than $0.1L$, they define

$$s(L) := \inf_{R \in \mathcal{R}_L^d} \inf_{\tau \in \Omega} \alpha(\mathcal{L}_R^\tau),$$

where we are optimizing over all possible rectangles of the same size and all possible boundary conditions. We will stress below the importance of optimizing over boundary conditions.

Next, they introduce a mixing condition on the Gibbs measure. Indeed, given Λ a rectangle of size L and $A, B \subset \Lambda$ of the same size and satisfying $A \cap B = \emptyset$, they assume that there exist constants $C_1, C_2 > 0$, depending on β, d and the commuting potential with respect to which the Hamiltonian and thus the Gibbs measure is defined, for which the following condition holds:

$$\sup_{\tau, \sigma \in \Omega} \left| \frac{\mu_\Lambda^\tau(\eta : \eta_A = \sigma_A) \mu_\Lambda^\tau(\eta : \eta_B = \sigma_B)}{\mu_\Lambda^\tau(\eta : \eta_{A \cup B} = \sigma_{A \cup B})} - 1 \right| \leq C_1 e^{-C_2 d(A, B)}. \quad (\text{Mix-Cond})$$

Assuming this condition to hold true, they prove the following reduction from rectangular lattices of size $2L$ to lattices of size L : There exists a positive constant k independent of L such that

$$s(2L) \geq \left(1 - \frac{k}{\sqrt{L}}\right) s(L). \quad (\text{Recurs})$$

This result allows for a recursion in L that implies the fact that

$$\inf_L s(L) > 0,$$

from which the positivity (and independence of Λ and τ) of the log-Sobolev constant follows immediately.

An essential point to prove (Recurs) is the fact that the optimization is also carried out on the boundary conditions, over which they average during the proof, and whose behaviour is “easily” controlled in the classical case due to the DLR conditions [Dob68] [LR69]. These conditions do not hold in the quantum case and thus we will have to introduce two new steps in our strategy with respect to the classical one.

Therefore, to sum up, we have seen that, in the classical case, a strategy consisting in three steps allows to obtain positivity of the log-Sobolev constant, under the assumption of a mixing condition (Mix-Cond) and after proving one result of quasi-factorization (QF) and a geometric recursion argument (Recurs).

One of the main objectives of this thesis is to provide a quantum analogue for this strategy, i.e., **a quantum strategy to prove positivity of log-Sobolev constants based on a result of quasi-factorization of the relative entropy**. This strategy will consist of five points, three of them being quantum versions of the three points mentioned above for the classical case, and two new ones that we have to introduce to compensate the lack of DLR conditions.

The strategy devised, whose graphical representation can be seen in the figure below, is the following one:

1. **Definition.** Definition of some *clustering conditions* on the Gibbs state.

This point is analogous to the use of the mixing condition (Mix-Cond) in the classical strategy. Throughout the manuscript, and depending on the system under study, we will introduce several notions of clustering of correlations on the fixed point (or set of fixed points) of the generator of the evolution. Most of the time, the role of this fixed point of the evolution will be played by a Gibbs state of a local, commuting Hamiltonian.

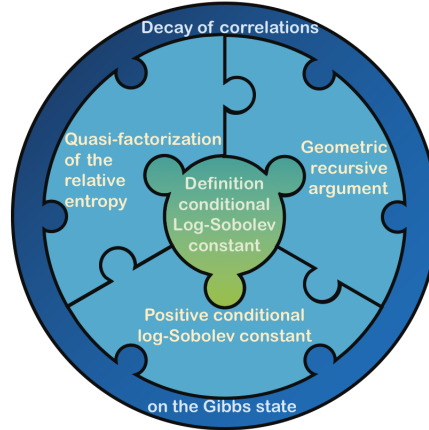


Figure 1.1: Complete puzzle to prove the positivity of a logarithmic Sobolev constant

2. **Definition.** *Definition of a conditional log-Sobolev constant.*

This is one of the new points. In the quantum setting, we need to introduce the definition of a conditional log-Sobolev constant from a conditional relative entropy, which will act as a quantum analogue of the classical conditional entropy mentioned above. More specifically, for every local generator \mathcal{L}_A^* in the Heisenberg picture on a (quantum) finite lattice Λ with fixed point σ_Λ , and given a subregion $A \subseteq \Lambda$, we will introduce the conditional log-Sobolev constant in A in the following way:

$$\alpha_\Lambda(\mathcal{L}_A^*) := \inf_{\rho_\Lambda \in \Lambda} \frac{-\text{tr}[\mathcal{L}_A^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_A(\rho_\Lambda || \sigma_\Lambda)},$$

where \mathcal{L}_A^* is a generator localized in A and $D_A(\rho_\Lambda || \sigma_\Lambda)$ is a conditional relative entropy, which we will have to introduce accordingly to each situation. Note that, in the classical case, this notion would agree with the log-Sobolev constant in A , due to the DLR conditions.

3. **Result.** *Quasi-factorization of the relative entropy in terms of a conditional relative entropy.*

This point constitutes the quantum analogue of the quasi-factorization for the entropy shown in (QF). It is clear that, to extend the classical result, first we need to introduce a suitable notion of conditional relative entropy (the same that for the conditional log-Sobolev constant) and, subsequently, given a tripartite space $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, we have to prove for every $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$ a result of the form:

$$D(\rho_{ABC} || \sigma_{ABC}) \leq \xi(\sigma_{ABC}) (D_{AB}(\rho_{ABC} || \sigma_{ABC}) + D_{BC}(\rho_{ABC} || \sigma_{ABC})),$$

where $\xi(\sigma_{ABC})$ reduces to the Radon-Nikodym derivative that appeared above when the states are classical and will provide some kind of measure of how far σ_{AC} is from being a tensor product between A and C .

We will also consider a strong version of this result, in which a conditional relative entropy appears in the LHS of the inequality instead, and will provide some examples of it.

4. **Result.** *Recursive geometric argument to reduce the global log-Sobolev constant to the conditional one in a fixed-sized region.*

In the classical version of this point, (Recurs), a recursive geometric argument is provided to reduce the value of the log-Sobolev constant in a big lattice to the log-Sobolev constant in a small one. In its quantum version, we will devise a recursive argument to reduce the value of a global log-Sobolev constant in a lattice to a conditional log-Sobolev constant in a subregion of it.

In some of the examples where this strategy is used throughout this text, the geometric argument will not be really recursive, since in those cases we will conceive a strategy that will allow us to execute this argument in just one step (see Chapters 9 and 10). However, we use this notation because, in the classical proof whose strategy we are extending here (see Chapter 3), there is indeed a recursion, as well as in some of the examples that appear in the quantum setting in the next chapters (see Chapter 11).

5. **Result.** *Positivity of the conditional log-Sobolev constant.*

To conclude, note that, as opposed to the classical case, we now need to prove the positivity of the conditional log-Sobolev constant to which we have reduced our global one in the previous step. In the classical case the positivity of the log-Sobolev constant in the small region was straightforward, as well as the independence with the size of Λ , but that is not granted in the quantum case anymore, as in the definition of that constant we are still optimizing over states defined on the whole Λ . Moreover, this will usually be the trickiest of the five points of the strategy.

Note that the first two points correspond to introducing some concepts in a suitable way, whereas the last three consist on proving certain results. Graphically, as shown in Figure 1.1, we could say that the strategy is composed of five different pieces, two of which we call *definition-pieces* and the other three *result-pieces* for obvious reasons, and we only obtain positivity of a log-Sobolev constant after having assembled all of them together.

Furthermore, the shape and location of the pieces in the puzzle is not arbitrary. Indeed, the step that lies at the core of this procedure is a correct definition of conditional log-Sobolev constant, without which it is impossible to continue the proof. Afterwards, we have to prove three different results that are equally important and which are strongly connected amongst themselves (the quasi-factorization is necessary to start the geometric recursive argument, which is useless unless one proves positivity of the conditional log-Sobolev constant, which at the same time motivates the result of quasi-factorization). Finally, all those results feel “incomplete” without the context given by the conditions of clustering of correlations, and this is the reason that this piece is located at the outer part of the puzzle.

1.3 MAIN RESULTS

In this section, we will briefly review the main results obtained in this thesis, all in the line of the strategy introduced above. After the introductory chapters, the text is split into three well-differentiated parts, each one of them corresponding to a different line of research (although all of them related to the core of the thesis); therefore, we will analyze each of these parts individually.

1.3.1 QUASI-FACTORIZATION OF THE RELATIVE ENTROPY

Part II is devoted to the study of results of quasi-factorization of the relative entropy, i.e. point (3) in the strategy mentioned in the previous section. However, before proving results of this kind, we need to introduce the notion of *conditional relative entropy*.

In Chapter 6, we define a conditional relative entropy as a function on pairs of quantum states satisfying a collection of axioms. More specifically, given a bipartite space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, the conditional relative entropy in A of two states in AB should provide the effect of the relative entropy of those states in the global space conditioned on the value of their relative entropy in B , extending the classical definition of conditional entropy of a function. Taking this into account, informally, a *conditional relative entropy* in A is defined as a function on pairs of quantum states such that:

- It is *continuous* with respect to the first variable.
- It is *non-negative* and vanishes if, and only if, both ρ_{AB} and σ_{AB} can be recovered by means of the Petz recovery map for the partial trace in A with respect to σ_{AB} .
- When considering the sum of the conditional relative entropies in A and B , it satisfies the properties of *additivity* and *superadditivity*.
- Concerning quantum channels, after adding the effect of the “B-part” of a channel to the conditional relative entropy in A , it satisfies a *data processing inequality*.

In principle, one could think that there exists a family of maps satisfying these properties. The surprise appears when we realize that actually there is only one possible map verifying them, and therefore they serve as an axiomatic characterization for the conditional relative entropy, which constitutes the first main result of this thesis.

Theorem 1.3.1 — AXIOMATIC CHARACTERIZATION OF THE CRE, (CLP18a).

Let $D_A(\cdot||\cdot)$ be a conditional relative entropy. Then, $D_A(\cdot||\cdot)$ is explicitly given by

$$D_A(\rho_{AB}||\sigma_{AB}) = D(\rho_{AB}||\sigma_{AB}) - D(\rho_B||\sigma_B),$$

for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$.

This notion is shown to extend its classical analogue presented above. Moreover, it allows us to pursue the quest of results of quasi-factorization of the relative entropy of the form stated in the strategy. Indeed, after imposing strong conditions on the states appearing in the relative entropies and obtaining some semi-trivial examples, in Chapter 7 we show the following result of quasi-factorization for the second state being a tensor product, which constitutes the first result of this kind and will serve as a basis to obtain results of positivity of log-Sobolev constants in the next part of the manuscript.

Theorem 1.3.2 — QUASI-FACTORIZATION FOR σ A TENSOR PRODUCT, (CLP18a).

Let \mathcal{H}_Λ be a multipartite Hilbert space and let $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$ such that $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$. The following inequality holds:

$$D(\rho_\Lambda||\sigma_\Lambda) \leq \sum_{x \in \Lambda} D_x(\rho_\Lambda||\sigma_\Lambda).$$

Note that in this result there is no multiplicative error term, since it should measure how far σ is from a tensor product and in this case it already satisfies that condition. Now, in Chapter 8, we take a further step in the complexity of these results, as we address the same problem for arbitrary ρ_{ABCD} in a 4-partite space and assume that σ_{ABCD} is a *quantum Markov chain* between $A \leftrightarrow C \leftrightarrow BD$, which means that

$$I_\sigma(A : BD|C) = 0,$$

where this quantity is called *conditional mutual information*. Some properties of quantum Markov chains will be introduced in Section 4.7, but for the time being let us just recall that the previous setting (for σ a tensor product) can be seen as a simplification of this one because these states can be split as a direct sum of tensor products in the “middle” system (in our case, C). For them, we prove the following result.

Theorem 1.3.3 — QUASI-FACTORIZATION FOR QUANTUM MARKOV CHAINS, (Bar+19).

Let $\mathcal{H}_{ABCD} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ be a 4-partite finite-dimensional Hilbert space, where system C shields A from B and D (in the sense that A and BD lie in different connected components of the system ABD), and let $\rho_{ABCD}, \sigma_{ABCD} \in \mathcal{S}_{ABCD}$. Let us further assume that σ_{ABCD} is a quantum Markov chain between $A \leftrightarrow C \leftrightarrow BD$. Then, the following inequality holds:

$$D_{AB}(\rho_{ABCD} || \sigma_{ABCD}) \leq D_A(\rho_{ABCD} || \sigma_{ABCD}) + D_B(\rho_{ABCD} || \sigma_{ABCD}). \quad (1.3)$$

Observe that, although σ_{ABCD} above is not a tensor product per se, there is no multiplicative error term either. Next, back to Chapter 7, we increase the difficulty by addressing the same problem for arbitrary states ρ_{ABC} and σ_{ABC} in a tripartite space. We obtain the following result, in which we observe that there is already a multiplicative error term with the desired meaning.

Theorem 1.3.4 — QUASI-FACTORIZATION FOR THE CRE, (CLP18a).

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ be a tripartite Hilbert space and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. Then, the following inequality holds

$$(1 - 2\|H(\sigma_{AC})\|_\infty)D(\rho_{ABC} || \sigma_{ABC}) \leq D_{AB}(\rho_{ABC} || \sigma_{ABC}) + D_{BC}(\rho_{ABC} || \sigma_{ABC}), \quad (1.4)$$

where

$$H(\sigma_{AC}) = \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \sigma_{AC} \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC}.$$

Note that $H(\sigma_{AC}) = 0$ if σ_{AC} is a tensor product between A and C .

Because of the form of the conditional relative entropy, this result of quasi-factorization can be equivalently phrased so that it constitutes an extension of the property of superadditivity of the relative entropy for general states (we do so in Chapter 5). Indeed, let us recall that the property of superadditivity of the relative entropy states that for two states ρ_{AB}, σ_{AB} in a bipartite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ such that $\sigma_{AB} = \sigma_A \otimes \sigma_B$, the following inequality holds:

$$D(\rho_{AB} || \sigma_A \otimes \sigma_B) \geq D(\rho_A || \sigma_A) + D(\rho_B || \sigma_B).$$

Moreover, as a consequence of the data processing inequality for the partial trace, the following inequality holds for every state ρ_{AB}, σ_{AB} :

$$2D(\rho_{AB} || \sigma_{AB}) \geq D(\rho_A || \sigma_A) + D(\rho_B || \sigma_B).$$

Therefore, the result below constitutes an extension to the property of superadditivity, since it holds for any possible σ_{AB} , not only tensor products, and provides a better multiplicative error term than the one obtained from the data processing inequality above, not only because it is tighter, but also because it measures how far σ_{AB} is from a tensor product.

Theorem 1.3.5 — SUPERADDITIVITY OF THE RELATIVE ENTROPY FOR GENERAL STATES, (CLP18b).

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite space. For any bipartite states $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$, the following inequality holds:

$$\min\{1 + 2\|H(\sigma_{AB})\|_\infty, 2\} D(\rho_{AB} || \sigma_{AB}) \geq D(\rho_A || \sigma_A) + D(\rho_B || \sigma_B),$$

where

$$H(\sigma_{AB}) = \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB},$$

and $\mathbb{1}_{AB}$ denotes the identity operator in \mathcal{H}_{AB} .

Note that $H(\sigma_{AB}) = 0$ if $\sigma_{AB} = \sigma_A \otimes \sigma_B$.

Coming back to the definition of conditional relative entropy, from which we prove our results of quasi-factorization of the relative entropy, if we analyze the different axioms from the definition, the last one (concerning quantum channels) seems the less natural one. Removing this axiom from the definition yields a new concept, which we call *modified conditional relative entropy*, and for which we present one example, that we call *conditional relative entropy by expectations* and is defined as

$$D_A^E(\rho_{AB} || \sigma_{AB}) := D(\rho_{AB} || \mathbb{E}_A^*(\rho_{AB})),$$

for all states ρ_{AB}, σ_{AB} in $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathbb{E}_A^*(\rho_{AB})$ coincides with the Petz recovery map for the partial trace, composed with the partial trace, i.e.

$$\mathbb{E}_A^*(\rho_{AB}) := \sigma_{AB}^{1/2} \sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \sigma_{AB}^{1/2}.$$

Considering this quantity, we aim to prove another result of quasi-factorization of the relative entropy for it, analogously to the result mentioned above for the conditional relative entropy. However, because of the form of this new kind of conditional relative entropy, we can prove a result of quasi-factorization of the relative entropy in a bipartite space, but the multiplicative error term we obtain, although having the same spirit than its analogue appearing in Theorem 1.3.4, has a much more complicated form.

Theorem 1.3.6 — QUASI-FACTORIZATION FOR THE CRE BY EXPECTATIONS, (CLP18a).

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite Hilbert space and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. Then, the following inequality holds

$$(1 - \xi(\sigma_{AB})) D(\rho_{AB} || \sigma_{AB}) \leq D_A^E(\rho_{AB} || \sigma_{AB}) + D_B^E(\rho_{AB} || \sigma_{AB}),$$

where

$$\xi(\sigma_{AB}) = 2(E_1(t) + E_2(t)),$$

and

$$E_1(t) = \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{-\frac{1+iit}{2}} \sigma_{AB}^{\frac{1-it}{2}} \sigma_A^{-\frac{1+iit}{2}} - \mathbb{1}_{AB} \right\|_\infty \left\| \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+iit}{2}} \sigma_B^{-1/2} \right\|_\infty,$$

$$E_2(t) = \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{-\frac{1-it}{2}} \sigma_{AB}^{\frac{1+iit}{2}} \sigma_A^{-\frac{1-it}{2}} - \mathbb{1}_{AB} \right\|_\infty,$$

with

$$\beta_0(t) = \frac{\pi}{2} (\cosh(\pi t) + 1)^{-1}.$$

Note that $\xi(\sigma_{AB}) = 0$ if σ_{AB} is a tensor product between A and B .

To conclude this part of the thesis, we now turn in Chapter 8 to a more abstract setting with the purpose of proving results of strong quasi-factorization of the relative entropy, i.e., in which there appears a conditional relative entropy in the left-hand side of the inequality of the quasi-factorization instead of a usual relative entropy. This kind of results allows for more freedom in the geometric resursive part of the strategy to prove positivity of the log-Sobolev constants, as we will discuss in Chapter 11.

The main difference with the former results of (weak) quasi-factorization lies in the fact that now we need to assume further conditions on σ , the second state appearing in the relative entropies, for these results to hold. Moreover, the result of strong quasi-factorization is proven for general conditional relative entropies by expectations, which are defined in the following way: Given a von Neumann algebra \mathcal{M} and $\mathcal{N} \subset \mathcal{M}$ a subalgebra of it, let σ be a state in \mathcal{M} and $\mathcal{E}_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ be the unique conditional expectation with respect to σ . Then, the *general conditional relative entropy by expectations* in \mathcal{N} is defined for every state ρ as

$$D_{\mathcal{N}}^{\mathcal{E}}(\rho || \sigma) := D(\rho || \mathcal{E}_{\mathcal{N}}^*(\rho)).$$

Before stating the main result of Chapter 8, let us introduce two conditions of clustering of correlations that will constitute assumptions for this result to hold.

First, given \mathcal{H} a finite-dimensional Hilbert space, \mathcal{N}_1 and \mathcal{N}_2 two von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$, $\mathcal{M} \subset \mathcal{N}_1 \cap \mathcal{N}_2$ another subalgebra, and a state σ , consider $\mathcal{E}_i : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{N}_i$ for $i = 1, 2$ and $\mathcal{E}_{\mathcal{M}} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{M}$ the unique conditional expectations on \mathcal{N}_i and \mathcal{M} with respect to σ , respectively. Then, we say that σ satisfies *conditional \mathbb{L}_1 -clustering of correlations* with respect to the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ if there exists a constant c such that the following holds for any $X \in \mathcal{B}(\mathcal{H})$:

$$|\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(X))| \leq c \|X\|_{\mathbb{L}_1(\sigma)}^2,$$

where the conditional covariance is given by

$$\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(X)) := \langle \mathcal{E}_1(X) - \mathcal{E}_{\mathcal{M}}(X), \mathcal{E}_2(X) - \mathcal{E}_{\mathcal{M}}(X) \rangle_{\sigma}.$$

Moreover, the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ satisfies *conditional \mathbb{L}_1 -clustering of correlations* if every state $\sigma = \mathcal{E}_{\mathcal{M}}^*(\sigma)$ satisfies it with the same constant c .

In the same conditions above, we further say that the state σ satisfies *covariance-entropy clustering of correlations* with respect to the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ if there exists a constant c such that the following holds for any $X \in \mathcal{B}(\mathcal{H})$:

$$|\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(X))| \leq c D(\Gamma_{\sigma}(X) || \Gamma_{\sigma} \circ \mathcal{E}_{\mathcal{M}}(X)),$$

where $\Gamma_{\sigma}(X) := \sigma^{1/2} X \sigma^{1/2}$.

Then, the main result of Chapter 8 is the following one.

Theorem 1.3.7 — STRONG QUASI-FACTORIZATION UNDER CONDITIONAL \mathbb{L}_1 -CLUSTERING OR COVARIANCE-ENTROPY CLUSTERING OF CORRELATIONS, (BCR19b).

Let \mathcal{H} be a finite-dimensional Hilbert space and let $\mathcal{N}_1, \mathcal{N}_2, \mathcal{M}$ be von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$ so that $\mathcal{M} \subset \mathcal{N}_1 \cap \mathcal{N}_2$. Let $\mathcal{E}_i : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{N}_i$, for $i = 1, 2$ and $\mathcal{E}_{\mathcal{M}} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{M}$ be conditional expectations with respect to a state σ .

Assume that there exists a constant $0 < c < \frac{1}{2(4+\sqrt{2})}$ such that the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ satisfies either conditional \mathbb{L}_1 -clustering of correlations or covariance-entropy clustering of correlations with corresponding constant c . Then, the following inequality holds for every $\rho \in \mathcal{S}(\mathcal{H})$:

$$D_{\mathcal{M}}^{\mathcal{E}}(\rho || \sigma) \leq \frac{1}{1 - 2(4+\sqrt{2})c} \left(D_1^{\mathcal{E}}(\rho || \sigma) + D_2^{\mathcal{E}}(\rho || \sigma) \right),$$

where $D_{\mathcal{M}}^{\mathcal{E}}(\rho || \sigma) := D(\rho || \mathcal{E}_{\mathcal{M}}^*(\rho))$ and $D_i^{\mathcal{E}}(\rho || \sigma) := D(\rho || \mathcal{E}_i^*(\rho))$ for $i = 1, 2$.

1.3.2 LOGARITHMIC SOBOLEV INEQUALITIES

In Part III, we focus on proving positivity of log-Sobolev constants for certain quantum dynamics. We address three different problems in three different chapters.

First, in Chapter 9 we consider the heat-bath dynamics with tensor product fixed point. More specifically, the global Lindbladian in this case is defined as the sum of local ones in the following form:

$$\mathcal{L}_{\Lambda}^* := \sum_{x \in \Lambda} \mathcal{L}_x^*,$$

where each \mathcal{L}_x^* is given by $\mathcal{L}_x^* := \mathbb{E}_x^* - \mathbb{1}_{\Lambda}$ for

$$\mathbb{E}_x^*(\rho_{\Lambda}) := \sigma_{\Lambda}^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_{\Lambda}^{1/2}$$

and the fixed point σ_{Λ} satisfies

$$\sigma_{\Lambda} = \bigotimes_{x \in \Lambda} \sigma_x.$$

Since σ_{Λ} is a product state, we can write $\mathbb{E}_x^*(\rho_{\Lambda})$ as

$$\mathbb{E}_x^*(\rho_{\Lambda}) = \sigma_x \otimes \rho_{x^c}.$$

Hence, for every $\rho_{\Lambda} \in \mathcal{S}_{\Lambda}$,

$$\mathcal{L}_{\Lambda}^*(\rho_{\Lambda}) = \sum_{x \in \Lambda} (\sigma_x \otimes \rho_{x^c} - \rho_{\Lambda}).$$

Then, for this Lindbladian and without any further assumption, following the steps presented in the strategy of Section 1.2, and using in particular Theorem 1.3.2, we prove the following result.

Theorem 1.3.8 — LOG-SOBOLEV CONSTANT FOR THE HEAT-BATH FOR TENSOR PRODUCTS, (CLP18a). \mathcal{L}_{Λ}^* defined as above has a global positive log-Sobolev constant.

Next, we consider in Chapter 10 again the heat-bath dynamics but now in 1D and assume weaker conditions on the fixed point. More specifically, given a finite chain $\Lambda \subset \mathbb{Z}$ and a state $\rho_{\Lambda} \in \mathcal{S}_{\Lambda}$, the heat-bath generator is defined as:

$$\mathcal{L}_{\Lambda}^*(\rho_{\Lambda}) = \sum_{x \in \Lambda} \left(\sigma_{\Lambda}^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_{\Lambda}^{1/2} - \rho_{\Lambda} \right),$$

where the first term in the sum of the RHS coincides with the Petz recovery map for the partial trace at every site $x \in \Lambda$, composed with the partial trace in x , and σ_Λ is the Gibbs state of a commuting k -local Hamiltonian.

We need to assume that a couple of clustering conditions on the Gibbs state hold. The first one is related to the exponential decay of correlations in the Gibbs state of the given commuting Hamiltonian and is satisfied, for example, by classical Gibbs states. Let $C, D \subset \Lambda$ be the union of non-overlapping finite-sized segments of Λ . The following inequality holds for positive constants K_1, K_2 independent of Λ :

$$\left\| \sigma_C^{-1/2} \otimes \sigma_D^{-1/2} \sigma_{CD} \sigma_C^{-1/2} \otimes \sigma_D^{-1/2} - \mathbb{1}_{CD} \right\|_\infty \leq K_1 e^{-K_2 d(C,D)},$$

where $d(C, D)$ is the distance between C and D , i.e., the minimum distance between two segments of C and D .

The second assumption constitutes a stronger form of quasi-factorization of the relative entropy than the ones mentioned above. An example where it holds is for Gibbs states verifying $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$. In words, given $X \subset \Lambda$, for every $\rho_\Lambda \in \mathcal{S}_\Lambda$ the following inequality holds

$$D_X(\rho_\Lambda \| \sigma_\Lambda) \leq f_X(\sigma_\Lambda) \sum_{x \in X} D_x(\rho_\Lambda \| \sigma_\Lambda),$$

where $1 \leq f_X(\sigma_\Lambda) < \infty$ depends only on σ_Λ and does not depend on the size of Λ , whereas $D_X(\rho_\Lambda \| \sigma_\Lambda)$, resp. $D_x(\rho_\Lambda \| \sigma_\Lambda)$, is the conditional relative entropy in X , resp. x , of ρ_Λ and σ_Λ .

Then, under these two assumptions, the following result is proven.

Theorem 1.3.9 — LOG-SOBOLEV CONSTANT FOR THE HEAT-BATH DYNAMICS IN 1D, (Bar+19).

Let $\Lambda \subset \mathbb{Z}$ be a finite chain. Let $\Phi : \Lambda \rightarrow \mathcal{A}_\Lambda$ be a k -local commuting potential, $H_\Lambda = \sum_{x \in \Lambda} \Phi(x)$ its corresponding Hamiltonian, and denote by σ_Λ its Gibbs state. Let \mathcal{L}_Λ^* be the generator of the heat-bath dynamics. Then, if both assumptions written above hold, the log-Sobolev constant of \mathcal{L}_Λ^* is strictly positive and independent of $|\Lambda|$.

Finally, to conclude this part, we move in Chapter 11 to the Davies dynamics. In this case, the Lindbladian $\mathcal{L}_\Lambda^\beta : \mathcal{A}_\Lambda \rightarrow \mathcal{A}_\Lambda$ associated to this dynamics for a certain finite inverse temperature β is of the following form:

$$\mathcal{L}_\Lambda^\beta(X) = i[H_\Lambda, X] + \sum_{k \in \Lambda} \mathcal{L}_k^\beta(X),$$

and, given $A \subset \Lambda$, the local generator is constructed by restricting the sum above to A :

$$\mathcal{L}_A^\beta(X) = i[H_A, X] + \sum_{k \in A} \mathcal{L}_k^\beta(X).$$

Then, we define the *conditional expectation* onto the algebra \mathcal{N}_A of fixed points of \mathcal{L}_A^β with respect to the Gibbs state σ_Λ^β as follows:

$$\mathcal{E}_A^\beta(X) := \lim_{t \rightarrow \infty} e^{t \mathcal{L}_A^\beta}(X).$$

Now, we can consider for this conditional expectation the respective definition of general conditional relative entropy by expectations, for which we showed a result of strong quasi-factorization of the relative entropy in Theorem 1.3.7. Assuming the same conditions of clustering of correlations that were needed there, and from this result of quasi-factorization, a geometric recursive argument in the line of the one showed for the classical case, and a conjecture on the positivity of the conditional log-Sobolev constant (which has already been communicated to us to be proven), the following result concerning the log-Sobolev constant of the Davies dynamics holds.

Theorem 1.3.10 — LOG-SOBOLEV CONSTANT FOR THE DAVIES DYNAMICS, (BCR19b).

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice and let β be a finite inverse temperature. Consider $\mathcal{L}_\Lambda^{\beta*} : \mathcal{S}_\Lambda \rightarrow \mathcal{S}_\Lambda$ the Lindbladian associated to the Davies dynamics and assume that either conditional \mathbb{L}_1 -clustering of correlations or covariance-entropy clustering of correlations is satisfied. Then, if Conjecture 11.3.1 holds true, $\mathcal{L}_\Lambda^{\beta*}$ has a positive log-Sobolev constant which is independent of $|\Lambda|$.

Note that there are several differences between the results on the log-Sobolev constants associated to the heat-bath and the Davies dynamics. The most remarkable one is the fact that the result for heat-bath only holds in 1D, whereas the result for the Davies holds for any finite dimension. This difference appears because of the different geometries that need to be considered for the geometric recursive argument to hold, which are devised in that way due to the fact that a result of strong quasi-factorization was proven for the Davies dynamics, whereas we only managed to prove a (weak) quasi-factorization for the heat-bath dynamics.

Moreover, the conditions assumed on the (set of) fixed points also differ, although all of them reduce to the same condition classically, the Dobrushin-Shlosman one, due to the DLR conditions. While the mixing condition assumed for the heat-bath dynamics looks more similar to the one assumed in the classical paper [DPP02], the ones conceived for the Davies generator are closer to those of [KB16].

1.3.3 DATA PROCESSING INEQUALITY FOR THE BS-ENTROPY

In the last part of the thesis, we turn to a more information-theoretical setting and study the data processing inequality for maximal f -divergences.

Quantum f -divergences are employed in quantum information theory to quantify the similarity of quantum states. The relative entropy is one example of the so-called *standard f -divergences* [HM17, Section 3.2], which are defined as

$$S_f(\sigma \parallel \rho) := \text{tr} \left[\rho^{1/2} f(L_\sigma R_{\rho^{-1}}) (\rho^{1/2}) \right]$$

for an operator convex function $f : (0, \infty) \rightarrow \mathbb{R}$. Here, L_A and R_A denote the left and right multiplication by the matrix A , respectively. The relative entropy arises by letting $f(x) = x \log x$.

This is, however, not the only way to generalize the classical f -divergences introduced in [AS66; Csi67]. The *maximal f -divergences* are defined as

$$\hat{S}_f(\sigma \parallel \rho) := \text{tr} \left[\rho f(\rho^{-1/2} \sigma \rho^{-1/2}) \right]$$

for an operator convex function $f : (0, \infty) \rightarrow \mathbb{R}$ and were defined in [PR98]. They were recently studied in [Mat10] where also the name was introduced (see also [HM17, Section 3.3]). For $f(x) = x \log x$, we obtain the relative entropy introduced by Belavkin and Staszewski in [BS82], which we will call *BS-entropy* for short:

$$\hat{S}_{\text{BS}}(\sigma \parallel \rho) := -\text{tr} \left[\sigma \log \left(\sigma^{-1/2} \rho \sigma^{-1/2} \right) \right].$$

Both the standard and maximal f -divergences satisfy data processing, i.e., given a quantum channel Φ , the following inequality holds for every state ρ and σ :

$$S_f(\sigma \parallel \rho) \geq S_f(\Phi(\sigma) \parallel \Phi(\rho)),$$

and analogously for maximal f -divergences. The study of conditions for equality in the previous inequality, and more specifically in the data processing inequality for the relative entropy, i.e. for

which ρ, σ we have

$$D(\sigma\|\rho) = D(\Phi(\sigma)\|\Phi(\rho))$$

for some fixed quantum channel Φ , has led to the discovery of quantum Markov states [Hay+04]. In particular, the relative entropy is preserved if and only if σ and ρ can be recovered by the Petz recovery map:

$$\mathcal{R}_\Phi^\rho(X) = \rho^{1/2} \Phi^*(\Phi(\rho)^{-1/2} X \Phi(\rho)^{-1/2}) \rho^{1/2},$$

i.e. $\sigma = \mathcal{R}_\Phi^\rho(\Phi(\sigma))$ and $\rho = \mathcal{R}_\Phi^\rho(\Phi(\rho))$ [Pet03]. This is true for all standard f -divergences for which f is “complicated enough”. We refer the reader to [HM17, Theorem 3.18] for a list of equivalent conditions.

For $\Phi = \mathcal{E}$ and \mathcal{E} the trace-preserving conditional expectation onto a unital matrix subalgebra \mathcal{N} of $\mathcal{B}(\mathcal{H})$, [CV17] shows that the equality condition is stable in the sense that the following inequality holds:

$$D(\sigma\|\rho) - D(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \geq \left(\frac{\pi}{8}\right)^4 \|L_\rho R_{\sigma^{-1}}\|_\infty^{-2} \|\mathcal{R}_\Phi^\sigma(\rho_{\mathcal{N}}) - \rho\|_1^4. \quad (1.5)$$

Here, we have written $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$ and $\rho_{\mathcal{N}} := \mathcal{E}(\rho)$. This can also be interpreted as a strengthening of the data processing inequality. Subsequent work has generalized the above result to more general standard f -divergences [CV18] and Holevo’s just-as-good fidelity [Wil18].

The difference of relative entropies that appears on the left hand-side of Equation (1.5) has been studied intensively in the context of quantum information and quantum thermodynamics [FBB18; FR18]. Moreover, for \mathcal{E} a partial trace, it has been characterized as a conditional relative entropy in [CLP18a] (see Chapter 6). Equation (1.5) is the first strengthening of the data processing inequality for the relative entropy in terms of the “distance” between a state and its recovery by the Petz map, although there have been many other results with a similar spirit in the last years.

In Chapter 12, we provide analogous results to the ones of [CV17] and [CV18] for maximal f -divergences. For these, preservation of the maximal f -divergence, i.e.

$$\hat{S}_f(\Phi(\sigma)\|\Phi(\rho)) = \hat{S}_f(\sigma\|\rho),$$

is not equivalent to σ, ρ being recoverable in the sense of Petz, although the latter implies the former. Equivalent conditions to the preservation of a maximal f -divergence for the case in which Φ is a completely positive trace-preserving map are given in [HM17, Theorem 3.34]. In Chapter 12, we prove two other equivalent conditions, which we use to prove a strengthened data processing inequality for some maximal f -divergences and in particular for the BS-entropy.

All quantum systems appearing in the text are finite dimensional. Let σ, ρ be two positive definite quantum states on a matrix algebra \mathcal{M} . We use the abbreviations $\Gamma := \sigma^{-1/2} \rho \sigma^{-1/2}$ and $\Gamma_{\mathcal{T}} := \sigma_{\mathcal{T}}^{-1/2} \rho_{\mathcal{T}} \sigma_{\mathcal{T}}^{-1/2}$, where \mathcal{N} is another matrix algebra, $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ is a completely positive trace-preserving map and $\rho_{\mathcal{T}} := \mathcal{T}(\rho)$, $\sigma_{\mathcal{T}} := \mathcal{T}(\sigma)$. Our first result consists of two conditions which are equivalent to the preservation of the BS-entropy under \mathcal{T} . It follows from Theorem 12.2.2 together with Proposition 12.2.5 and Theorem 12.5.1.

Theorem 1.3.11 — A CONDITION FOR EQUALITY IN THE DPI FOR THE BS ENTROPY, (BC19b).

Let \mathcal{M} and \mathcal{N} be two matrix algebras and let $\sigma > 0$, $\rho > 0$ be two quantum states on \mathcal{M} . Let $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ be a completely positive trace-preserving map and let V be the isometry associated to the Stinespring dilation (Theorem 4.4.9) of \mathcal{T} . Then, the following are equivalent:

1. $\hat{S}_{\text{BS}}(\sigma \| \rho) = \hat{S}_{\text{BS}}(\sigma_{\mathcal{T}} \| \rho_{\mathcal{T}})$
2. $\sigma^{-1} \rho = \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}})$
3. $V \sigma^{1/2} V^* \left(\sigma_{\mathcal{T}}^{-1/2} \Gamma_{\mathcal{T}}^{1/2} \sigma_{\mathcal{T}}^{1/2} \otimes I \right) = V \Gamma^{1/2} \sigma^{1/2} V^*.$

The above theorem is motivated by the treatment in [Pet03] for the relative entropy and proceeds along similar lines. This result allows us to prove a strengthened data processing inequality for the BS-entropy, building on the work in [CV17] for conditional expectations and subsequently lifting it to general quantum channels using Stinespring's dilation theorem:

Theorem 1.3.12 — STRENGTHENED DPI FOR THE BS-ENTROPY, (BC19b).

Let \mathcal{M} and \mathcal{N} be two matrix algebras and let $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ be a completely positive trace-preserving map. Let σ, ρ be two quantum states on \mathcal{M} such that they have equal support. Then,

$$\hat{S}_{\text{BS}}(\sigma \| \rho) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{T}} \| \rho_{\mathcal{T}}) \geq \left(\frac{\pi}{8} \right)^4 \|\Gamma\|_{\infty}^{-4} \|\sigma_{\mathcal{T}}^{-1}\|_{\infty}^{-2} \|\sigma \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}}) - \rho\|_2^4. \quad (1.6)$$

Theorem 1.3.11 shows that the right hand side of Equation (1.6) plays the same role as the trace distance between ρ and the state obtained from the recovery map in Equation (1.5). The result for conditional expectations appears as Corollary 12.3.5 in the main text and follows from the sharper lower bound in Theorem 12.3.3. These results are subsequently lifted to general quantum channels in Theorem 12.5.1.

In the rest of the work, we extend the result from the BS-entropy to more general maximal f -divergences. This is similar to the work undertaken in [CV18]. We consider operator convex functions $f : (0, \infty) \rightarrow \mathbb{R}$ whose transpose $\tilde{f}(x) := xf(1/x)$ is operator monotone decreasing. Moreover, we assume that the measure $\mu_{-\tilde{f}}$ of $-\tilde{f}$ is absolutely continuous with respect to Lebesgue measure and that there are $C > 0$, $\alpha \geq 0$ such that for every $T \geq 1$, the Radon-Nikodým derivative is lower bounded by

$$\frac{d\mu_{-\tilde{f}}(t)}{dt} \geq (CT^{2\alpha})^{-1}$$

almost everywhere (with respect to Lebesgue measure) for all $t \in [1/T, T]$. Furthermore, we assume that our states $\sigma > 0$, $\rho > 0$ are not too far from fulfilling the data processing inequality with respect to \mathcal{E} , i.e.

$$\left(\frac{(2\alpha + 1)\sqrt{C}}{4} \frac{(\hat{S}_f(\sigma \| \rho) - \hat{S}_f(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}}))^{1/2}}{1 + \|\Gamma\|_{\infty}} \right)^{\frac{1}{1+\alpha}} \leq 1. \quad (1.7)$$

Theorem 1.3.13 — STABILITY OF THE DPI FOR MAXIMAL f -DIVERGENCES, (BC19b).

Let \mathcal{M} and \mathcal{N} be two matrix algebras and let $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ be a completely positive trace-preserving map. Let σ, ρ be two quantum states on \mathcal{M} such that they have equal support and let $f : (0, \infty) \rightarrow \mathbb{R}$ be an operator convex function with transpose \tilde{f} . We assume that \tilde{f} is operator monotone decreasing and such that the measure $\mu_{-\tilde{f}}$ that appears in Theorem 4.4.2 is absolutely continuous with respect to Lebesgue measure. Moreover, we assume that for every $T \geq 1$, there exist constants $\alpha \geq 0, C > 0$ satisfying $d\mu_{-\tilde{f}}(t)/dt \geq (CT^{2\alpha})^{-1}$ for all $t \in [1/T, T]$ and such that Equation (1.7) holds. Then, there is a constant $L_\alpha > 0$ such that

$$\begin{aligned} \hat{S}_f(\sigma \parallel \rho) - \hat{S}_f(\sigma_{\mathcal{T}} \parallel \rho_{\mathcal{T}}) \\ \geq \frac{L_\alpha}{C} (1 + \|\Gamma\|_\infty)^{-(4\alpha+2)} \|\Gamma\|_\infty^{-(2\alpha+2)} \|\sigma_{\mathcal{T}}^{-1}\|_\infty^{-(2\alpha+2)} \|\rho - \sigma \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}})\|_2^{4(\alpha+1)}. \end{aligned}$$

For conditional expectations, the above result appears as Corollary 12.4.2 in the main text and follows from the sharper lower bound in Theorem 12.4.1. The extension to general quantum channels appears as Theorem 12.5.3.

1.4 ORGANIZATION OF THE THESIS

The contents of this thesis are organized as follows. In Chapter 1, we provide an introduction to the problems addressed in this thesis, the results that have been proven and settle some notation (a translated version of this chapter to Spanish is Chapter 2). Subsequently, in Chapter 3, we introduce classical spin systems, discuss the classical analogous problem of proving positivity of log-Sobolev constants and elaborate on the strategy employed for this result which is based on results of quasi-factorization of the entropy. We conclude the introductory part of the thesis by reviewing several preliminary notions and properties that will be necessary to understand the rest of the text in Chapter 4.

In Part II, we focus on results of quasi-factorization of the relative entropy. First, in Chapter 5, we present a quantitative extension for the property of superadditivity of the relative entropy for general states. After introducing and characterizing several concepts of conditional relative entropy in Chapter 6, we show some results of quasi-factorization of the relative entropy for different conditional relative entropies in Chapter 7. Subsequently, we present some stronger versions of these results of quasi-factorization in Chapter 8.

We turn to the study of logarithmic Sobolev inequalities in Part III. This study begins with the particular case of a tensor product as fixed point of the evolution corresponding to the heat-bath dynamics in Chapter 9, for which we show that the log-Sobolev constant is always lower bounded by $1/2$. Next, we consider again the heat-bath dynamics, but assume weaker conditions on the fixed point of the evolution and show in Chapter 10 that, if it corresponds to the Gibbs state of a local commuting Hamiltonian, under two assumption of clustering of correlations in this state, the log-Sobolev constant associated to 1D systems is positive. To conclude this part, we turn to Davies dynamics in Chapter 11, for which we address the problem of proving positivity of log-Sobolev constants under certain conditions of clustering of correlations, via the results of strong quasi-factorization mentioned above.

Finally, in Part IV, and more specifically Chapter 12, we address the problem of strengthening the data processing inequality associated to the BS-entropy. First, we provide two new conditions that are equivalent to having equality in the data processing inequality for the BS-entropy, which allows to define a BS-recovery condition. Subsequently we use this conditions to provide a strengthened version of the data processing inequality for the BS-entropy and, with more generality, for a big class of maximal f -divergences.

To conclude, the main results of this thesis have been communicated in the following scientific publications:

- (CLP18b) Á. Capel, A. Lucia and D. Pérez-García, ***Superadditivity of Quantum Relative Entropy for General States***, *IEEE Trans. on Inf. Theory*, 64 (7) (2018), 4758-4765, DOI: [10.1109/TIT.2017.2772800](https://doi.org/10.1109/TIT.2017.2772800), arXiv: [1705.03521](https://arxiv.org/abs/1705.03521), (Chapter 5).
- (CLP18a) Á. Capel, A. Lucia and D. Pérez-García, ***Quantum Conditional Relative Entropy and Quasi-Factorization of the Relative Entropy***, *J. Phys. A: Math. Theor.*, 51 (2018), 484001, DOI: [10.1088/1751-8121/aae4cf](https://doi.org/10.1088/1751-8121/aae4cf), arXiv: [1804.09525](https://arxiv.org/abs/1804.09525), (Chapters 6, 7 and 9).
- (BC19b) A. Bluhm and Á. Capel, ***A strengthened data processing inequality for the Belavkin-Staszewski relative entropy***, *Rev. Math. Phys.*, to appear (2019), DOI: [10.1142/S0129055X20500051](https://doi.org/10.1142/S0129055X20500051), arXiv: [1904.10768](https://arxiv.org/abs/1904.10768), (Chapter 12).
- (Bar+19) I. Bardet, Á. Capel, A. Lucia, D. Pérez-García and C. Rouzé, ***On the modified logarithmic Sobolev inequality for the heat-bath dynamics for 1D systems***, preprint (2019), arXiv: [1908.09004](https://arxiv.org/abs/1908.09004), (Chapter 10 and Section 8.2).
- (BCR19b) I. Bardet, Á. Capel and C. Rouzé, ***Positivity of the modified logarithmic Sobolev constant for quantum Davies semigroups: the commuting case***, in preparation (2019), (Chapters 8 and 11).

Another result in a different research line that the candidate has obtained during the PhD, which is not included in the core of the thesis to homogenize as much as possible, but will be briefly discussed in Appendix 12.5 is based on the following article:

- (CMM17) Á. Capel, M. Martín and J. Merí, ***Numerical radius attaining compact linear operators***, *J. Math. Anal. Appl.*, 445 (2017), 1258-1266, DOI: [10.1016/j.jmaa.2016.02.074](https://doi.org/10.1016/j.jmaa.2016.02.074), arXiv: [1602.07084](https://arxiv.org/abs/1602.07084).



2. INTRODUCCIÓN

2.1 NOTACIÓN Y ANTECEDENTES

2.1.1 NOTACIÓN

Antes de comenzar, fijemos algunas notaciones que serán empleadas a lo largo de esta tesis, aunque algunos objetos concretos serán presentados con más detalle después.

En este texto, consideramos espacios de Hilbert complejos finito dimensionales. Para Λ un conjunto constituido por $|\Lambda|$ partes, denotamos al espacio de Hilbert multipartito finito dimensional de $|\Lambda|$ partes por:

$$\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x,$$

donde \mathcal{H}_x es un espacio de Hilbert finito dimensional asociado a cada sitio x de la retícula. Denotaremos por $|\psi\rangle$ a los vectores de \mathcal{H}_x y por $\langle\psi|$ a sus adjuntos.

A lo largo de este texto, frecuentemente Λ constará de tres partes, y denotaremos por $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ al correspondiente espacio de Hilbert tripartito. Además, una gran parte de esta tesis trata sobre sistemas reticulares de espines cuánticos, por lo que frecuentemente consideraremos a $\Lambda \subset \mathbb{Z}^d$ como un subconjunto finito. En general, usamos letras latinas mayúsculas para representar sistemas o conjuntos.

Para cada \mathcal{H}_Λ finito dimensional, denotamos al conjunto de operadores lineales y acotados en él por $\mathcal{B}_\Lambda := \mathcal{B}(\mathcal{H}_\Lambda)$, y por $\mathcal{A}_\Lambda := \mathcal{A}(\mathcal{H}_\Lambda)$ a su subconjunto de *observables*, es decir, operadores Hermíticos, los cuales denotamos por letras latinas minúsculas. Más aún, denotamos por

$$\mathcal{S}_\Lambda := \mathcal{S}(\mathcal{H}_\Lambda) = \{f_\Lambda \in \mathcal{A}_\Lambda : f_\Lambda \geq 0 \text{ and } \text{tr}[f_\Lambda] = 1\}$$

al conjunto de *matrices de densidad*, o *estados*, y escribimos sus elementos empleando letras minúsculas griegas. En particular, siempre que aparezcan en el texto, denotaremos a los *estados de Gibbs* por σ_Λ . Frecuentemente escribiremos el espacio en el que cada operador está definido usando el mismo subíndice que para el espacio, pero puede que lo omitamos cuando sea innecesario.

This is an image of Palma de Mallorca (Spain) during the 3rd edition of *Quantum Information in Spain ICE-3*, in April 2016, the first workshop that I attended in Spain during the thesis.

Escribimos $\mathbb{1}$ para la matriz identidad e id para el operador identidad. Para espacios bipartitos $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, consideramos la inclusión natural $\mathcal{A}_A \hookrightarrow \mathcal{A}_{AB}$ que se obtiene al identificar cada operador $f_A \in \mathcal{A}_A$ con $f_A \otimes \mathbb{1}_B$.

Dado un espacio de Hilbert bipartito $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, definimos la *traza parcial sobre A* como la única aplicación $\text{tr}_A : \mathcal{B}_{AB} \rightarrow \mathcal{B}_B$ tal que $\text{tr}_A[a \otimes b] = b \text{tr}[a]$ para todo $a \in \mathcal{B}_A$ y $b \in \mathcal{B}_B$. Además, definimos la traza parcial modificada en A de $f_{AB} \in \mathcal{A}_{AB}$ como $\text{tr}_A[f_{AB}] \otimes \mathbb{1}_B$, pero lo denotamos por $\text{tr}_A[f_{AB}]$ en un pequeño abuso de notación. Además, diremos que un operador $g_{AB} \in \mathcal{A}_{AB}$ tiene soporte en A si se puede escribir como $g_A \otimes \mathbb{1}_B$ para algún operador $g_A \in \mathcal{A}_A$. Remarcamos en este punto que dado $f_{AB} \in \mathcal{A}_{AB}$, escribimos entonces $f_A := \text{tr}_B[f_{AB}]$.

Un *canal cuántico* [Wol12] es una aplicación completamente positiva y que preserva la traza. Llamamos a un operador lineal $\mathcal{T} : \mathcal{B}_\Lambda \rightarrow \mathcal{B}_\Lambda$ un *superoperador*, y decimos que es *positivo* si lleva operadores positivos a operadores positivos. Además, decimos que \mathcal{T} es *completamente positivo* si, dado \mathcal{M}_n el espacio de matrices complejas $n \times n$, se tiene que $\mathcal{T} \otimes \text{id} : \mathcal{B}_\Lambda \otimes \mathcal{M}_n \rightarrow \mathcal{B}_\Lambda \otimes \mathcal{M}_n$ es positivo para cada $n \in \mathbb{N}$. Finalmente, decimos que \mathcal{T} *preserva la traza* si $\text{tr}[\mathcal{T}(f_\Lambda)] = \text{tr}[f_\Lambda]$ para cada $f_\Lambda \in \mathcal{B}_\Lambda$.

Dadas \mathcal{M} y \mathcal{N} dos álgebras de von Neumann, si consideramos un operador $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$, denotamos por $\mathcal{T}^* : \mathcal{N} \rightarrow \mathcal{M}$ a su aplicación dual. Además, frecuentemente consideraremos \mathcal{B}_Λ equipado con el *producto interno de Hilbert-Schmidt*, es decir,

$$\langle A, B \rangle_{\text{HS}} := \text{tr}[A^* B] \quad \text{for } A, B \in \mathcal{B}_\Lambda,$$

donde A^* representa al adjunto de A. Algunas veces, consideraremos también el *producto interno con peso σ* dado por

$$\langle A, B \rangle_\sigma := \text{tr}[\sigma^{1/2} A^* \sigma^{1/2} B] \quad \text{for } A, B \in \mathcal{B}_\Lambda.$$

En general, denotaremos por $\|\cdot\|_{\mathbb{L}_p(\sigma)}$ a las *normas \mathbb{L}_p con peso σ* , y por $\|\cdot\|_p$ a las *p-normas de Schatten*, dadas por $\text{tr}[|\cdot|^p]^{1/p}$. En particular, escribimos $\|\cdot\|_\infty$ para la norma de operadores usual, así como $\|\cdot\|_1 = \text{tr}[|\cdot|]$ para la norma tracial.

Dados $x, y \in \Lambda \subset \mathbb{Z}^d$, denotamos por $d(x, y)$ a la distancia euclídea entre x e y en \mathbb{Z}^d . Por tanto, la distancia entre dos subconjuntos de Λ , A y B , viene dada por

$$d(A, B) := \min\{d(x, y) : x \in A, y \in B\}.$$

Finalmente, comentemos brevemente el hecho de que nos restrinjamos a espacios de Hilbert finito dimensionales. En numerosos sistemas cuánticos, los observables son operadores en un espacio de Hilbert \mathcal{H} infinito dimensional, pero separable. Se puede extender fácilmente la definición de la traza, y por tanto de las matrices de densidad, a este escenario infinito dimensional. Sin embargo, es relativamente sencillo mostrar que cualquier operador de densidad en \mathcal{H} (es decir, un operador semidefinido positivo y con traza 1) es un operador compacto, y, por tanto, puesto que la propiedad de aproximación se satisface en esta clase de espacios, se puede aproximar por operadores de rango finito en la norma de operadores. Aunque este razonamiento diste de implicar que los resultados en espacios de Hilbert finito dimensionales se puedan extender a espacios infinito dimensionales de forma directa, al menos proporciona un argumento para resaltar que los aspectos iniciales de las desigualdades para matrices de densidad que aparecerán a lo largo del texto se observan ya de por sí en el caso finito dimensional (ver [Car09]).

2.1.2 EVOLUCIONES DISIPATIVAS CUÁNTICAS

Los postulados de la mecánica cuántica afirman que un sistema físico aislado viene completamente descrito por un operador de densidad en un espacio de Hilbert complejo que es conocido

como el espacio de estados del sistema. Además, la evolución del sistema aislado se describe a través de transformaciones unitarias, es decir, si las propiedades físicas del sistema están codificadas en el operador de densidad ρ , entonces la evolución del sistema viene dada por $U\rho U^*$, con U un operador unitario. Esta evolución es claramente reversible y su inversa viene dada por $U^*\rho U$.

Sin embargo, esta clase teórica de sistemas no constituye un enfoque adecuado para modelizar y estudiar situaciones realistas relativas a sistemas microscópicos, ya que cualquier sistema cuántico de muchos cuerpos se encuentra contenido en un baño térmico enorme, que es extremadamente complicado de modelizar y con el que existen interacciones no despreciables. Por tanto, ningún experimento se puede realizar a temperatura cero o completamente aislado del ruido. Entonces, para tener un enfoque más realista, tenemos que centrarnos en sistemas cuánticos abiertos, esto es sistemas de muchos cuerpos que están rodeados por un entorno con el que interactúan de forma no despreciable. La dinámica resultante es por tanto disipativa y, en particular, irreversible.

Describamos ahora cómo es la evolución de un sistema cuántico de muchos cuerpos abierto. Asumamos que \mathcal{H} es el espacio de Hilbert finito asociado a un cierto sistema cuántico y consideremos la aplicación $\Phi: \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{S}(\mathcal{H})$ que describe su evolución. Podemos justificar de dos formas diferentes que Φ tiene que ser un canal cuántico, teniendo en cuenta las propiedades que una evolución físicamente realizable debería satisfacer.

Primero, Φ debería verificar las siguientes propiedades:

- Debería enviar estados a estados, lo cual implica que tiene que ser lineal, positiva y preservar la traza.
- Si consideramos $\rho \in \mathcal{S}(\mathcal{H})$ y $\sigma \in \mathcal{S}(\mathcal{H}')$ en un espacio de Hilbert diferente con evolución trivial, la aplicación compuesta tiene que verificar:

$$\begin{aligned} \hat{\Phi}: \mathcal{S}(\mathcal{H} \otimes \mathcal{H}') &\rightarrow \mathcal{S}(\mathcal{H} \otimes \mathcal{H}') \\ \rho \otimes \sigma &\mapsto \Phi(\rho) \otimes \sigma, \end{aligned}$$

donde $\hat{\Phi} = \Phi \otimes \text{id}$ debería ser positivo.

Puesto que esto se debería cumplir para \mathcal{H}' de cualquier dimensión, $\hat{\Phi}$ es completamente positivo.

Juntando todas estas propiedades, concluimos que $\hat{\Phi}$ es un canal cuántico.

Otra forma de justificar este hecho es la siguiente. Aunque nos estemos centrando ahora en sistemas cuánticos abiertos, que consisten en sistemas que interactúan con un entorno, el par sistema-entorno sí que constituye un sistema cerrado. Por tanto, ambos juntos sí que evolucionan a partir de un operador unitario, como hemos mencionado antes.

Más específicamente, si asumimos que el estado asociado al entorno es un estado puro, $|\psi\rangle\langle\psi|_E$, entonces la evolución de un estado inicial ρ en el sistema viene dada por:

$$\rho \mapsto \rho \otimes |\psi\rangle\langle\psi|_E \mapsto U(\rho \otimes |\psi\rangle\langle\psi|_E)U^* \mapsto \text{tr}_E[U(\rho \otimes |\psi\rangle\langle\psi|_E)U^*] \equiv \tilde{\rho},$$

donde trazamos el entorno, tr_E , para obtener el efecto en el sistema. Es fácil darse cuenta de que cada uno de los pasos que aparecen arriba constituye un canal cuántico, y por tanto la evolución del sistema abierto es descrita por un canal cuántico. Más información sobre el hecho de que todo canal cuántico se pueda ver como la restricción de una evolución a través de una unitaria aparecerá en el Capítulo 12, donde usaremos este hecho para construir resultados para canales cuánticos desde esperanzas condicionadas usando el teorema de dilatación de Stinespring.

El razonamiento previo es válido para cada paso de la evolución dinámica. Para su descripción continua en el tiempo, tenemos que fijarnos en que, para cada instante $t \geq 0$, la correspondiente rebanada de tiempo es una evolución realizable \mathcal{T}_t y por tanto un canal cuántico.

Para construir la evolución continua en el tiempo desde esto, consideramos que el efecto del entorno en el sistema es casi irrelevante, pero tiene que ser tenido en cuenta. Sin embargo, debido al pequeño efecto provocado en el sistema, podemos asumir que el entorno no evoluciona con el tiempo, lo cual es algo razonable, por ejemplo, si el entorno es un baño térmico que es mucho más grande que el sistema y cuya temperatura va a ser prácticamente invariante.

Por este motivo, podemos asumir el *límite de acoplamiento débil*, que significa que en la evolución del par sistema-entorno mencionada antes se puede considerar que $U(\rho \otimes |\psi\rangle\langle\psi|_E)U^*$ está muy cerca de $\tilde{\rho} \otimes |\psi\rangle\langle\psi|_E$ para un cierto $\tilde{\rho}$.

Además, vamos a reducir nuestro estudio al caso en el que el entorno no tiene memoria, por lo que la evolución futura solo depende del presente. Esto es conocido como la *aproximación Markoviana*.

Asumiendo todas estas condiciones, podemos definir una *evolución disipativa cuántica* (o *semigrupo de Markov cuántico*) como un semigrupo continuo uniparamétrico $\{\mathcal{T}_t^*\}_{t \geq 0}$ de aplicaciones completamente positivas y que preservan la traza verificando:

- $\mathcal{T}_0^* = \mathbb{1}$.
- $\mathcal{T}_t^* \circ \mathcal{T}_s^* = \mathcal{T}_{t+s}^*$,

para cada $t, s \geq 0$, donde estamos empleando la notación $*$ para enfatizar el hecho de que nos encontramos en el enfoque de Schrödinger.

Dada una retícula finita $\Lambda \subset \mathbb{Z}^d$, denotamos al generador de este semigrupo por \mathcal{L}_Λ^* y lo llamamos *Lindbladiano* (or *Liouvillian*), puesto que su versión dual en el enfoque de Heisenberg satisface la forma de Lindblad (o GKLS) [Lin76], [GKS76] para cada $X_\Lambda \in \mathcal{B}_\Lambda$:

$$\mathcal{L}_\Lambda(X_\Lambda) = i[H, X_\Lambda] + \frac{1}{2} \sum_{k=1}^l [2L_k^* X_\Lambda L_k - (L_k^* L_k X_\Lambda + X_\Lambda L_k^* L_k)],$$

donde $H \in \mathcal{A}_\Lambda$, los $L_k \in \mathcal{B}_\Lambda$ son los *operadores de Lindblad* y $[\cdot, \cdot]$ representa al conmutador. Además, se le llama Liouvillian por satisfacer la ecuación de Liouville, es decir:

$$\frac{d}{dt} \mathcal{T}_t^* = \mathcal{L}_\Lambda^* \circ \mathcal{T}_t^* = \mathcal{T}_t^* \circ \mathcal{L}_\Lambda^*.$$

Por tanto, podemos escribir los elementos del semigrupo de Markov cuántico como sigue:

$$\mathcal{T}_t^* = e^{t\mathcal{L}_\Lambda^*}.$$

2.1.3 TIEMPO DE EQUILIBRACIÓN Y DESIGUALDADES DE LOG-SOBOLEV

Consideremos de nuevo la retícula Λ mencionada antes. Dado $\rho_\Lambda \in \mathcal{S}_\Lambda$, escribimos

$$\rho_t := \mathcal{T}_t^*(\rho_\Lambda)$$

para cada $t \geq 0$ (cuando el subíndice se pueda omitir sin causar ninguna confusión). Con esta notación, está claro que la evolución del sistema se puede describir a través de la siguiente ecuación dinámica cuántica:

$$\partial_t \rho_t = \mathcal{L}_\Lambda^*(\rho_t).$$

Decimos que un cierto estado σ_Λ es un *estado invariante* de $\{\mathcal{T}_t^*\}_{t \geq 0}$ si

$$\sigma_t := \mathcal{T}_t^*(\sigma_\Lambda) = \sigma_\Lambda$$

para todo $t \geq 0$.

A lo largo de esta tesis, nos restringiremos al caso *primitivo*, es decir, en el que $\{\mathcal{T}_t^*\}_{t \geq 0}$ tiene un único estado invariante de rango máximo (y por tanto hay un único σ_Λ para el que $\mathcal{L}_\Lambda^*(\sigma_\Lambda) = 0$). Asumamos también que el proceso de Markov cuántico estudiado es *reversible*, es decir, que satisface la *condición de equilibrio detallado*

$$\langle f, \mathcal{L}_\Lambda(g) \rangle_{\sigma_\Lambda} = \langle \mathcal{L}_\Lambda(f), g \rangle_{\sigma_\Lambda}$$

para cada $f, g \in \mathcal{A}_\Lambda$, donde \mathcal{L}_Λ es el generador del semigrupo de evolución en el enfoque de Heisenberg. Un problema interesante relativo a los semigrupos de Markov cuánticos es el estudio de la velocidad de convergencia hacia este único estado invariante, que se puede hacer acotando el tiempo de equilibración.

El *tiempo de equilibración* de un semigrupo cuántico de Markov es el tiempo que tarda un cierto estado inicial en convertirse casi indistinguible del estado invariante, es decir, del punto fijo de la evolución. Más específicamente, para cada $\varepsilon > 0$, viene dado por la siguiente expresión

$$\tau(\varepsilon) := \min \left\{ t > 0 : \sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \|\rho_t - \sigma_\Lambda\|_1 \leq \varepsilon \right\}. \quad (2.1)$$

A partir de este concepto, existe una clase especial de generadores para los que esta convergencia es “suficientemente rápida”, lo cual llamamos *equilibración rápida*. En detalle, decimos que \mathcal{L}_Λ^* satisface *equilibración rápida* si

$$\sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \|\rho_t - \sigma_\Lambda\|_1 \leq \text{poly}(|\Lambda|) e^{-\gamma t},$$

para una constante $\gamma > 0$ y donde $\text{poly}(|\Lambda|)$ se refiere a un polinomio en $|\Lambda|$. Esta propiedad tiene profundas implicaciones en el sistema, como la estabilidad frente a perturbaciones externas [Cub+15] y el hecho de que sus puntos fijos satisfagan una ley de área para la información mutua [Bra+15a].

Como ya hemos mencionado, un problema fundamental que se encuentra en la intersección entre la teoría de la información cuántica y la física de la materia condensada es la velocidad de convergencia de este tipo de evoluciones disipativas a sus estados de equilibrio, o puntos fijos, que se puede hacer estudiando el tiempo de equilibración de la evolución. Diferentes cotas para el tiempo de equilibración se pueden obtener a partir de las constantes óptimas para ciertas desigualdades funcionales cuánticas, tales como el *gap espectral* para la *desigualdad de Poincaré* [Tem+10] y la *constante logarítmica de Sobolev* para la *desigualdad logarítmica de Sobolev* [KT16]. En esta tesis nos centraremos en la última.

Existe toda una familia de *desigualdades logarítmicas de Sobolev* (*desigualdades de log-Sobolev* en corto), que se puede indexar con un parámetro entero, como se hizo en [KT16]. Este texto trata la desigualdad 1-log-Sobolev, también conocida en la literatura como la desigualdad de log-Sobolev modificada (MLSI) y que llamaremos en este texto simplemente *desigualdad de log-Sobolev* (con su respectiva *constante de log-Sobolev* asociada), puesto que no hay posible confusión. Es importante remarcar, sin embargo, que este concepto no se corresponde clásicamente con la constante de log-Sobolev clásica, aunque la constante 2-log-Sobolev sí lo hace.

En detalle, dado un Lindbladiano reversible, primitivo $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \rightarrow \mathcal{S}_\Lambda$ con punto fijo $\sigma_\Lambda \in \mathcal{S}_\Lambda$, definimos la *constante de log-Sobolev* de \mathcal{L}_Λ^* como

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\text{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)},$$

donde $D(\rho_\Lambda || \sigma_\Lambda)$ es la *entropía relativa* entre ρ_Λ y σ_Λ y viene dada por

$$D(\rho_\Lambda || \sigma_\Lambda) := \text{tr}[\rho_\Lambda(\log \rho_\Lambda - \log \sigma_\Lambda)].$$

La constante de log-Sobolev, cuando es positiva, proporciona una cota superior para el tiempo de equilibración de una evolución disipativa. La derivación de la cota que se obtiene

para el tiempo de equilibración en función de las constantes de log-Sobolev se puede encontrar explícitamente en la Sección 4.5, y da lugar a la expresión:

$$\|\rho_t - \sigma_\Lambda\|_1 \leq \sqrt{2\log(1/\sigma_{\min})} e^{-\alpha(\mathcal{L}_\Lambda^*)t}, \quad (2.2)$$

Esta cota proporciona una mejora exponencial con respecto a la que se obtiene en función del gap espectral (ver discusión sobre este tema en [KT16]). Además, en el mismo artículo los autores demostraron que la última implica la primera, es decir, que si un sistema tiene una constante de log-Sobolev positiva, entonces también tiene un gap espectral positivo. Por tanto, es razonable plantearse el problema de encontrar sistemas que tienen una constante de log-Sobolev positiva de entre los que se sabe que tienen un gap espectral positivo.

Probar si un Lindbladiano tiene constante de log-Sobolev positiva es, por tanto, un problema fundamental en sistemas cuánticos de muchos cuerpos abiertos. In esta línea, el principal propósito de esta tesis es:

Encontrar condiciones estáticas en el punto fijo de una evolución disipativa que sean condición suficiente para que el sistema tenga una constante de log-Sobolev positiva.

Esto, a su vez, proporcionaría condiciones bajo las cuales un sistema satisfaría equilibración rápida, una propiedad con numerosas implicaciones en teoría de la información cuántica, como ya hemos mencionado previamente.

Este problema ya se ha estudiado con anterioridad en sistemas clásicos. Como veremos después, en [DPP02] se demostró que un sistema de espines clásicos en una retícula, para una cierta dinámica y una condición de agrupamiento en la medida de Gibbs asociada a esta dinámica, tiene una constante de log-Sobolev positiva. Este resultado, inspirado por el gran trabajo de Martinelli and Olivieri [MO94a], simplificó notablemente su prueba utilizando resultados de quasi-factorización de la entropía. Previamente, un resultado de quasi-factorización de la varianza [BCC02] se había utilizado para probar positividad del gap espectral para algunas dinámicas.

Este último resultado encontró su análogo cuántico en [KB16], donde los autores probaron positividad del gap espectral para las dinámicas de Davies y heat-bath asociadas a un Hamiltoniano conmutante local, a través de un resultado de quasi-factorización de la varianza, bajo una condición de agrupamiento fuerte en el estado de Gibbs. Estos generadores constituyen clases de muestreadores de Gibbs in el campo de sistemas cuánticos, que se emplean para desarrollar algoritmos de simulación y muestreo, los cuales se pueden utilizar para preparar grandes clases de estados térmicos de Hamiltonianos físicamente relevantes. Más específicamente, el generador de Davies se deriva del acoplamiento débil de un sistema cuántico finito con un baño térmico grande, mientras que el generador de heat-bath se construye siguiendo la misma idea que para el algoritmo clásico de Monte-Carlo de heat-bath.

Por tanto, gracias a los resultados positivos obtenidos para el gap espectral para estas dos clases de dinámicas, sólo trataremos en este texto el problema de probar positividad de la constante de log-Sobolev para las dinámicas de heat-bath y Davies.

Siguiendo el propósito introducido arriba, podemos enunciar ahora los dos principales objetivos de la tesis como sigue:

1. **Desarrollar una estrategia para probar que un sistema cuántico tiene una constante de log-Sobolev positiva, a partir de resultados de quasi-factorización de la entropía relativa.**
2. **Aplicar dicha estrategia a las dinámicas de heat-bath y Davies, bajo ciertas condiciones en los puntos fijos de las evoluciones.**

En la próxima sección, nos centraremos en el primero de estos objetivos. Más específicamente, a partir de los resultados para sistemas de espines clásicos, diseñaremos e implementaremos una estrategia para probar que un sistema cuántico tiene una constante de log-Sobolev

positiva basada en varios pasos, algunos de ellos relativos a encontrar definiciones adecuadas para ciertos conceptos, y otros que consisten en algunos resultados que necesitamos probar.

Posteriormente, en la Sección 2.3, comentaremos los principales resultados obtenidos en la tesis en la línea del segundo objetivo presentado antes, es decir, resultados en la línea de probar positividad de constantes de log-Sobolev para las dinámicas de heat-bath o Davies basados en la estrategia previamente presentada.

2.2 ESTRATEGIA PARA ENCONTRAR CONSTANTES DE LOG-SOBOLEV POSITIVAS

El problema de probar si un cierto sistema tiene constante de log-Sobolev positiva ya se estudió previamente para sistemas de espines clásicos. En [DPP02], los autores demostraron que un sistema de espines clásicos en una retícula, para una cierta dinámica y bajo ciertas condiciones de agrupamiento de la medida de Gibbs asociada a esta dinámica, satisface una *desigualdad logarítmica de Sobolev modificada*, o *desigualdad de entropía*, cuyo análogo cuántico llamamos en este texto simplemente *desigualdad de log-Sobolev*.

En [Ces01], la *desigualdad logarítmica de Sobolev* usual, correspondiente a la *desigualdad 2-log-Sobolev* en el caso cuántico, fue estudiada a partir de otras condiciones similares de agrupamiento en la medida de Gibbs. Ambos resultados vinieron inspirados por el gran trabajo de Martinelli y Olivieri [Mar99], [MO94a], [MO94b] con el propósito de simplificar notablemente su prueba utilizando un resultado de quasi-factorización de la entropía en función de algunas entropías condicionadas. Previamente, un resultado de quasi-factorización de la varianza [BCC02] fue empleado para probar la positividad del gap espectral para ciertas dinámicas, bajo ciertas condiciones en la medida de Gibbs.

Fijémonos ahora en el principal resultado de [DPP02] mencionado anteriormente, y más específicamente en la estrategia que se siguió en él. Este resultado se discutirá en detalle en el Capítulo 3. Por el momento, introduzcamos simplemente algunas nociones básicas y expliquemos brevemente los diferentes pasos seguidos en la prueba.

Consideremos un espacio de probabilidad $(\Omega, \mathcal{F}, \mu)$. Para cada f en Ω con $f > 0$, la *entropía* de f se define como

$$\text{Ent}_\mu(f) := \mu(f \log f) - \mu(f) \log \mu(f).$$

Además, considerando que \mathcal{L}_Λ^τ es el generador de Markov de la dinámica estocástica estudiada en [DPP02], para una retícula finita $\Lambda \subset \subset \mathbb{Z}^d$ y una condición de frontera $\tau \in \Omega$, la *forma de Dirichlet* asociada a \mathcal{L}_Λ^τ viene dada por

$$\mathcal{E}_\Lambda^\tau(f, g) := -\mu_\Lambda^\tau(f \mathcal{L}_\Lambda^\tau g),$$

donde μ_Λ^τ es la *medida de Gibbs* en Λ con condición de frontera τ , que corresponde a la única medida invariante para la dinámica, y cuyo análogo cuántico en nuestro resultado será el estado de Gibbs.

En ese caso, podemos definir la *constante de log-Sobolev* (que aparece como *constante entrópica* en [DPP02]) como

$$\alpha(\mathcal{L}_\Lambda^\tau) := \inf \left\{ \frac{\mathcal{E}_\Lambda^\tau(f, \log f)}{\text{Ent}_{\mu_\Lambda^\tau}(f)} : f \geq 0, f \log f \in \mathcal{L}_1(\mu_\Lambda^\tau), \text{Ent}_{\mu_\Lambda^\tau}(f) \neq 0 \right\}. \quad (\text{Log-Sob})$$

Ahora, de vuelta en espacios de probabilidad $(\Omega, \mathcal{F}, \mu)$ más generales y dada una sub- σ -álgebra $\mathcal{G} \subseteq \mathcal{F}$, se define una *entropía conditional* en \mathcal{G} de la siguiente manera para cada $f > 0$:

$$\text{Ent}_\mu(f | \mathcal{G}) := \mu(f \log f | \mathcal{G}) - \mu(f | \mathcal{G}) \log \mu(f | \mathcal{G}),$$

donde $\mu(f \mid \mathcal{G})$ viene dado por

$$\int_G \mu(f \mid \mathcal{G}) d\mu = \int_G f d\mu \quad \text{para cada } G \in \mathcal{G}.$$

Con estas definiciones, primero prueban un resultado de quasi-factorización de la entropía. En detalle, dados $\mathcal{F}_1, \mathcal{F}_2$ sub- σ -álgebras de \mathcal{F} , y asumiendo que existe una probabilidad $\bar{\mu}$ que hace a \mathcal{F}_1 y \mathcal{F}_2 independientes, y para la que $\mu \ll \bar{\mu}$ y $\mu \mid \mathcal{F}_i = \bar{\mu} \mid \mathcal{F}_i$ para $i = 1, 2$, se prueba que para cada $f \geq 0$ tal que $f \log f \in L^1(\mu)$ y $\mu(f) = 1$ se tiene la siguiente desigualdad:

$$\text{Ent}_\mu(f) \leq \frac{1}{1 - 4\|h - 1\|_\infty} \mu [\text{Ent}_\mu(f \mid \mathcal{F}_1) + \text{Ent}_\mu(f \mid \mathcal{F}_2)], \quad (\text{QF})$$

donde $h = \frac{d\mu}{d\bar{\mu}}$ es la derivada de Radon-Nikodym de μ con respecto a $\bar{\mu}$, y por tanto mide en un cierto sentido cómo de lejos está μ de hacer a \mathcal{F}_1 y \mathcal{F}_2 independientes.

Posteriormente, dada una retícula inicial $\Lambda \subset \mathbb{Z}^d$, diseñan una cierta partición geométrica para Λ en función de algunas subregiones solapadas que permite reducir la constante de log-Sobolev en Λ a la constante de log-Sobolev en una subregión de tamaño pequeño. Más específicamente, dada una familia de subregiones de Λ rectangulares d -dimensionales cuyo mayor lado tiene tamaño L y cuyo menor lado no es más pequeño que $0.1L$, definen

$$s(L) := \inf_{R \in \mathcal{R}_L^d} \inf_{\tau \in \Omega} \alpha(\mathcal{L}_R^\tau),$$

donde estamos optimizando sobre todos los posibles rectángulos del mismo tamaño y todas las posibles condiciones de frontera. Más abajo resaltaremos la importancia de optimizar sobre las condiciones de frontera.

Ahora, introducimos una condición de equilibración en la medida de Gibbs. En detalle, dado Λ un rectángulo de tamaño L y $A, B \subset \Lambda$ del mismo tamaño y que satisfacen $A \cap B = \emptyset$, asumimos que existen $C_1, C_2 > 0$, dependiendo de β, d y del potencial conmutante con respecto al cual se definen el Hamiltonian y la medida de Gibbs, para las que se tiene la siguiente condición:

$$\sup_{\tau, \sigma \in \Omega} \left| \frac{\mu_\Lambda^\tau(\eta : \eta_A = \sigma_A) \mu_\Lambda^\tau(\eta : \eta_B = \sigma_B)}{\mu_\Lambda^\tau(\eta : \eta_{A \cup B} = \sigma_{A \cup B})} - 1 \right| \leq C_1 e^{-C_2 d(A, B)}. \quad (\text{Mix-Cond})$$

Asumiendo que esta condición se tiene, en [DPP02] se prueba la siguiente reducción de retículas rectangulares de tamaño $2L$ a retículas de tamaño L : Existe una constante positiva k , independiente de L , tal que

$$s(2L) \geq \left(1 - \frac{k}{\sqrt{L}}\right) s(L). \quad (\text{Recurs})$$

Este resultado da lugar a una recursión en L que implica el siguiente hecho:

$$\inf_L s(L) > 0,$$

a partir del cual inmediatamente se sigue la positividad (y la independencia de Λ y τ) de la constante de log-Sobolev.

Un punto esencial para probar (Recurs) es el hecho de que la optimización se lleve a cabo también sobre las condiciones de frontera, sobre las cuales promedian durante la prueba, y cuyo comportamiento es “fácilmente” controlado en el caso clásico debido a las condiciones DLR [Dob68] [LR69]. Estas condiciones no se cumplen en el caso cuántico y por tanto tendremos que añadir dos pasos nuevos a nuestra estrategia con respecto a la clásica.

En consecuencia, para resumir, hemos visto que, en el caso clásico, una estrategia consistente en tres pasos permite obtener positividad para la constante de log-Sobolev bajo la suposición de una condición de equilibración (Mix-Cond) y tras probar un resultado de quasi-factorización (QF) y un argumento geométrico recursivo (Recurs).

Uno de los principales objetivos de esta tesis es proporcionar un análogo cuántico, es decir, **una estrategia cuántica para probar positividad de constantes de log-Sobolev basada en un resultado de quasi-factorización de la entropía relativa**. Esta estrategia constará de cinco puntos, tres de los cuales constituyen versiones cuánticas de los tres puntos mencionados anteriormente en el caso clásico, y dos nuevos que hemos añadido para compensar la carencia de las condiciones DLR.

La estrategia diseñada, cuya representación gráfica se puede ver en la figura de abajo, es la siguiente:

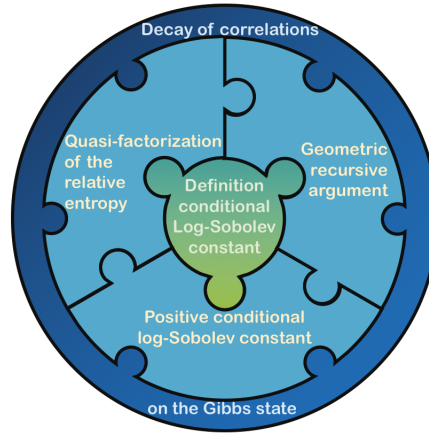


Figure 2.1: Puzzle completo para probar positividad de una constante logarítmica de Sobolev.

1. **Definición.** Definición de algunas *condiciones de agrupamiento en el estado de Gibbs*.

Este punto es el análogo al uso de las condiciones de equilibración (Mix-Cond) en la estrategia clásica. A lo largo de este texto, y dependiendo en el sistema de estudio, introduciremos varias nociones de agrupamiento de correlaciones en el punto fijo (o conjunto de puntos fijos) del generador. La mayor parte del tiempo, el papel de este punto fijo de la evolución lo jugará el estado de Gibbs de un Hamiltoniano conmutante local.

2. **Definición.** Definición de una *constante de log-Sobolev condicionada*.

Este es uno de los nuevos puntos. En el caso cuántico, necesitamos introducir la definición de una constante de log-Sobolev condicionada a partir de una entropía relativa condicionada, un análogo cuántico a la entropía condicionada clásica mencionada anteriormente. Más específicamente, para cada generador local \mathcal{L}_A^* en el enfoque de Heisenberg en una retícula (cuántica) finita Λ con punto fijo σ_Λ , y dada una subregión $A \subseteq \Lambda$, introduciremos la constante de log-Sobolev condicionada en A de la siguiente manera:

$$\alpha_\Lambda(\mathcal{L}_A^*) := \inf_{\rho_\Lambda \in \Lambda} \frac{-\text{tr}[\mathcal{L}_A^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_A(\rho_\Lambda || \sigma_\Lambda)},$$

donde \mathcal{L}_A^* es un generador localizado en A y $D_A(\rho_\Lambda || \sigma_\Lambda)$ es una entropía relativa condicionada, que tendremos que definir adecuadamente en cada situación. Resaltemos que, en

el caso clásico, este concepto coincide con la constante de log-Sobolev en A , gracias a las condiciones DLR.

3. **Resultado.** *Quasi-factorización de la entropía relativa en función de entropías relativas condicionadas.*

Este punto constituye el análogo cuántico de la quasi-factorización de la entropía mostrada en (QF). Es claro que, para extender el resultado clásico, primero necesitamos introducir una noción adecuada de entropía relativa condicionada (la misma que para la constante de log-Sobolev condicionada) y, posteriormente, dado un espacio tripartito $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, tenemos que probar para cada $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$ un resultado de la forma:

$$D(\rho_{ABC} || \sigma_{ABC}) \leq \xi(\sigma_{ABC}) (D_{AB}(\rho_{ABC} || \sigma_{ABC}) + D_{BC}(\rho_{ABC} || \sigma_{ABC})),$$

donde $\xi(\sigma_{ABC})$ se reduzca a la derivada de Radon-Nikodym que aparecía anteriormente cuando los estados sean clásicos y proporcione algún tipo de medida de cómo de lejos se encuentra σ_{AC} de ser un producto tensor entre A y C .

También consideraremos una versión fuerte de este resultado, en la que una entropía relativa condicionada aparece en la parte de la izquierda de la desigualdad, en lugar de una entropía relativa, y proporcionaremos algunos ejemplos para ello.

4. **Resultado.** *Argumento geométrico recursivo para reducir la constante de log-Sobolev global a la condicionada en una región de tamaño fijo.*

En la versión clásica de este punto, (Recurs), se da un argumento geométrico recursivo para reducir el valor de la constante de log-Sobolev de una gran retícula a la constante de log-Sobolev en una pequeña. En esta versión cuántica, diseñaremos un argumento recursivo para reducir el valor de la constante de log-Sobolev global en una retícula a la constante de log-Sobolev condicionada en una subregión.

En algunos de los ejemplos de uso de esta estrategia a lo largo del texto, el argumento geométrico no será realmente recursivo, puesto que en esos casos desarrollaremos una estrategia que permitirá ejecutar este argumento en solo un paso (ver Capítulos 9 y 10). Sin embargo, usamos esta notación, puesto que en la prueba clásica cuya estrategia estamos extendiendo (ver Capítulo 3) hay de hecho una recursión, al igual que en algunos de los ejemplos que aparecen en el contexto cuántico en los siguientes capítulos (ver Capítulo 11).

5. **Resultado.** *Positividad de la constante de log-Sobolev condicionada.*

Para concluir, es importante resaltar que, opuestamente al caso clásico, ahora necesitamos probar la positividad de la constante de log-Sobolev condicionada a la que hemos reducido la global en el paso previo. En el caso clásico, la positividad de la constante de log-Sobolev en la región pequeña era directa, así como la independencia con el tamaño de Λ , pero esto no está garantizado en el caso cuántico, puesto que en la definición de dicha constante todavía estamos optimizando sobre estados definidos en toda Λ . Más aún, esta será normalmente la parte más complicada de los cinco puntos de la estrategia a estudiar.

Los dos primeros puntos corresponden a introducir ciertos conceptos de forma adecuada, mientras que los tres últimos consisten en probar algunos resultados. Gráficamente, como vimos en la Figura 1.1, se podría decir que la estrategia se compone de cinco puntos diferentes, dos de los cuales llamamos *piezas-definición* y los otros tres *piezas-resultado* por razones obvias, y solo obtenemos positividad de la constante de log-Sobolev tras haberlas encajado todas.

Además, la forma y ubicación de cada una de las piezas en el puzzle no es arbitraria. De

hecho, el paso que se encuentra en el núcleo de este procedimiento es la correcta definición de la constante de log-Sobolev condicionada, sin la cual es imposible continuar la prueba. Posteriormente, tenemos que probar tres resultados diferentes cuya importancia es equivalente y los cuales están fuertemente conectados entre ellos (la quasi-factorización es necesaria para comenzar el argumento geométrico recursivo, el cual es inútil a no ser que se pruebe la positividad de la constante de log-Sobolev condicionada, que al mismo tiempo motiva el resultado de quasi-factorización). Finalmente, todos estos resultados parecen estar “incompletos” en un cierto sentido sin el contexto proporcionado por las condiciones de agrupamiento de correlaciones, y esta es la razón de que esta pieza se encuentre localizada en la parte más externa del puzzle.

2.3 RESULTADOS PRINCIPALES

En esta sección, repasaremos brevemente los principales resultados obtenidos en esta tesis, todos en la línea de la estrategia anteriormente introducida. Tras los capítulos introductorios, el texto se divide en tres partes bien diferenciadas, cada una de las cuales corresponde a una línea diferente de investigación (aunque todas ellas relacionadas con el núcleo de la tesis); por tanto, analizaremos cada una de estas partes individualmente.

2.3.1 QUASI-FACTORIZACIÓN DE LA ENTROPÍA RELATIVA

La Parte II está dedicada al estudio de resultados de quasi-factorización de la entropía relativa, es decir, al punto (3) de la estrategia introducida en la sección anterior. Sin embargo, antes de probar resultados de este tipo, necesitamos introducir la noción de *entropía relativa condicionada*.

En el Capítulo 6, definimos una entropía relativa condicionada como una función sobre pares de estados cuánticos que satisface una serie de axiomas. Más específicamente, dado un espacio bipartito $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, la entropía relativa condicionada en A de dos estados en AB debería proporcionar el efecto de la entropía relativa de dichos estados en el espacio global condicionada al valor de su entropía relativa en B , extendiendo la definición clásica de entropía condicionada de una función. Teniendo esto en cuenta, informalmente, una *entropía relativa condicionada* (ERC) en A se define como una función sobre pares de estados cuánticos tal que:

- Es *continua* con respecto a la primera variable.
- Es *no negativa* y se anula si, y sólo si, ambos ρ_{AB} y σ_{AB} se pueden recuperar a través de la aplicación de recuperación de Petz para la traza parcial en A con respecto a σ_{AB} .
- Cuando consideramos la suma de las entropías relativas condicionadas en A y B , satisface las propiedades de *aditividad* y *superaditividad*.
- Relativo a canales cuánticos, tras añadir el efecto de la “parte B” de un canal a la entropía relativa condicionada en A , se satisface una *desigualdad de procesamiento de datos*.

En principio, se podría pensar que existe toda una familia de aplicaciones satisfaciendo estas propiedades. La sorpresa aparece al comprobar que, de hecho, sólo hay una posible aplicación verificándolas, y por tanto sirven como una caracterización axiomática de la entropía relativa condicionada, lo cual constituye el primer resultado principal de esta tesis.

Teorema 2.3.1 — CARACTERIZACIÓN AXIOMÁTICA DE LA ERC, (CLP18a).

Sea $D_A(\cdot||\cdot)$ una entropía relativa condicionada. Entonces, $D_A(\cdot||\cdot)$ viene dado explícitamente por

$$D_A(\rho_{AB}||\sigma_{AB}) = D(\rho_{AB}||\sigma_{AB}) - D(\rho_B||\sigma_B),$$

para cada $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$.

Se puede ver que este concepto extiende a su análogo clásico presentado antes. Además, permite perseguir la búsqueda de resultados de quasi-factorización de la entropía relativa de la forma presentada en la estrategia. De hecho, tras imponer condiciones fuertes en los estados que aparecen en las entropías relativas y obtener algunos ejemplos semi-triviales, en el Capítulo 7 demostramos el siguiente resultado de quasi-factorización para el caso en el que el segundo estado es un producto tensor, el cual constituye el primer resultado de este tipo y servirá como base para obtener resultados de positividad de constantes de log-Sobolev en la siguiente parte del texto.

Teorema 2.3.2 — QUASI-FACTORIZACIÓN PARA σ UN PRODUCTO TENSOR, (CLP18a).

Sea \mathcal{H}_Λ un espacio de Hilbert multipartito y sean $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$ tales que $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$. La siguiente desigualdad se cumple:

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda).$$

Cabe remarcar que en este resultado no hay término de error multiplicativo, puesto que éste debería medir cómo de lejos está σ de ser un producto tensor y en este caso ya satisface dicha condición de por sí. A continuación, en el Capítulo 8, damos un paso más en la complejidad en estos resultados, ya que consideramos el mismo problema para estados ρ_{ABCD} arbitrarios en espacios 4-partitos y asumimos que σ_{ABCD} es una *cadena de Markov cuántica* entre $A \leftrightarrow C \leftrightarrow BD$, lo cual quiere decir que:

$$I_\sigma(A : BD|C) = 0,$$

donde esta cantidad se llama *información mutua condicionada*. Se mostrarán algunas propiedades de cadenas de Markov cuánticas en la Sección 4.7, pero por el momento solo recordaremos que el escenario anterior (para σ un producto tensor) se puede ver como una simplificación de éste puesto que estos estados se pueden expresar como una suma directa de productos tensores, “partiendo” el sistema “intermedio” (en nuestro caso, C). Para ellos, probamos el siguiente resultado.

Teorema 2.3.3 — QUASI-FACTORIZACIÓN PARA CADENAS DE MARKOV CUÁNTICAS, (Bar+19).

Sea $\mathcal{H}_{ABCD} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ un espacio de Hilbert finito-dimensional 4-partito, donde el sistema C separa a A de B y D (en el sentido de que A y BD están en diferentes componentes conexas del sistema ABD), y sean $\rho_{ABCD}, \sigma_{ABCD} \in \mathcal{S}_{ABCD}$. Asumamos además que σ_{ABCD} es una cadena de Markov cuántica entre $A \leftrightarrow C \leftrightarrow BD$. Entonces, se tiene la siguiente desigualdad:

$$D_{AB}(\rho_{ABCD} || \sigma_{ABCD}) \leq D_A(\rho_{ABCD} || \sigma_{ABCD}) + D_B(\rho_{ABCD} || \sigma_{ABCD}). \quad (2.3)$$

Observemos que, aunque σ_{ABCD} no es un producto tensor per se, tampoco hay término de error multiplicativo. Ahora, de nuevo en el Capítulo 7, aumentamos la dificultad al plantearnos el mismo problema para estados arbitrarios ρ_{ABC} y σ_{ABC} en un espacio tripartito. Obtenemos el siguiente resultado, en el que observamos que sí que hay un término de error multiplicativo con el significado deseado.

Teorema 2.3.4 — QUASI-FACTORIZACIÓN PARA LA ERC, (CLP18a).

Sea $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ un espacio de Hilbert tripartito y sean $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. Se tiene la siguiente desigualdad

$$(1 - 2\|H(\sigma_{AC})\|_\infty)D(\rho_{ABC}||\sigma_{ABC}) \leq D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}), \quad (2.4)$$

donde

$$H(\sigma_{AC}) = \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \sigma_{AC} \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC}.$$

Observamos que $H(\sigma_{AC}) = 0$ si σ_{AC} es un producto tensor entre A y C .

Teniendo en cuenta la forma particular de la entropía relativa condicionada, este resultado de quasi-factorización puede ser equivalentemente formulado de forma que constituya una extensión de la propiedad de superaditividad de la entropía relativa para estados generales (lo hacemos en el Capítulo 5). De hecho, recordemos que la propiedad de superaditividad de la entropía relativa dice que para dos estados ρ_{AB}, σ_{AB} en un sistema bipartito $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ tal que $\sigma_{AB} = \sigma_A \otimes \sigma_B$, se tiene la siguiente desigualdad:

$$D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

Además, como consecuencia de la desigualdad de procesamiento de datos para la traza parcial, la siguiente desigualdad se tiene para todos los estados ρ_{AB}, σ_{AB} :

$$2D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

Por tanto, el siguiente resultado constituye una extensión a la propiedad de superaditividad, puesto que es válido para cada σ_{AB} posible, no solo para productos tensores, y da lugar a un mejor término de error multiplicativo que el obtenido para la desigualdad de procesamiento de datos anterior, no sólo por ser más fino, sino también porque mide cómo de lejos está σ_{AB} de ser un producto tensor.

Teorema 2.3.5 — SUPERADITIVIDAD DE LA ENTROPÍA RELATIVA PARA ESTADOS GENERALES, (CLP18b).

Sea $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ un espacio bipartito. Para cualesquiera estados bipartitos $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$, se tiene la siguiente desigualdad:

$$\min\{1 + 2\|H(\sigma_{AB})\|_\infty, 2\}D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B),$$

donde

$$H(\sigma_{AB}) = \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB},$$

y $\mathbb{1}_{AB}$ denota al operador identidad en \mathcal{H}_{AB} .

Observamos que $H(\sigma_{AB}) = 0$ si $\sigma_{AB} = \sigma_A \otimes \sigma_B$.

Volviendo a la definición de entropía relativa condicionada, a partir de la cual probamos nuestros resultados de quasi-factorización de la entropía relativa, si analizamos los diferentes axiomas de la definición, el último de ellos (el relativo a canales cuánticos) parece ser el menos natural. Quitar este axioma de la definición proporciona un nuevo concepto, que llamamos *entropía relativa condicionada modificada*, y para el cual presentamos un ejemplo, que llamamos *entropía relativa condicionada por esperanzas* y está definida como

$$D_A^E(\rho_{AB}||\sigma_{AB}) := D(\rho_{AB}||\mathbb{E}_A^*(\rho_{AB})),$$

para todos los estados ρ_{AB}, σ_{AB} en $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, donde $\mathbb{E}_A^*(\rho_{AB})$ coincide con la aplicación de recuperación de Petz para la traza parcial, compuesta con la traza parcial, es decir,

$$\mathbb{E}_A^*(\rho_{AB}) := \sigma_{AB}^{1/2} \sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \sigma_{AB}^{1/2}.$$

Considerando esta cantidad, nos planteamos el problema de probar otro resultado de quasi-factorización de la entropía relativa para ella, análogamente a lo realizado anteriormente para la entropía relativa condicionada. Sin embargo, por la forma que toma este nuevo tipo de entropía relativa condicionada, podemos probar un resultado de quasi-factorización de la entropía relativa en un espacio bipartito, pero el término de error multiplicativo que obtenemos, aunque vaya en la misma dirección que su análogo del Teorema 2.3.4, toma una forma mucho más complicada.

Teorema 2.3.6 — QUASI-FACTORIZACIÓN PARA LA ERC POR ESPERANZAS, (CLP18a).

Sea $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ un espacio de Hilbert bipartito y sean $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. Se tiene la siguiente desigualdad:

$$(1 - \xi(\sigma_{AB}))D(\rho_{AB}||\sigma_{AB}) \leq D_A^E(\rho_{AB}||\sigma_{AB}) + D_B^E(\rho_{AB}||\sigma_{AB}),$$

donde

$$\xi(\sigma_{AB}) = 2(E_1(t) + E_2(t)),$$

y

$$E_1(t) = \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{-\frac{1+it}{2}} \sigma_{AB}^{\frac{1-it}{2}} \sigma_A^{-\frac{1+it}{2}} - \mathbb{1}_{AB} \right\|_{\infty} \left\| \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+it}{2}} \sigma_B^{-1/2} \right\|_{\infty},$$

$$E_2(t) = \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{-\frac{1-it}{2}} \sigma_{AB}^{\frac{1+it}{2}} \sigma_A^{-\frac{1-it}{2}} - \mathbb{1}_{AB} \right\|_{\infty},$$

con

$$\beta_0(t) = \frac{\pi}{2} (\cosh(\pi t) + 1)^{-1}.$$

Observamos que $\xi(\sigma_{AB}) = 0$ si σ_{AB} es un producto tensor entre A y B .

Para concluir esta parte de la tesis, ahora cambiamos en el Capítulo 8 a un ambiente más abstracto con el propósito de probar resultados de quasi-factorización fuerte de la entropía relativa, es decir, en los que aparece una entropía relativa condicionada en la parte de la izquierda de la desigualdad en lugar de una entropía relativa usual. Este tipo de resultados permite una mayor libertad en la parte geométrica recursiva de la estrategia para probar positividad de constantes de log-Sobolev, como comentaremos en el Capítulo 11.

La principal diferencia con los anteriores resultados de quasi-factorización (débil) se encuentra en el hecho de que ahora necesitamos asumir más condiciones en σ , el segundo estado que aparece en las entropías relativas, para que el resultado sea cierto. Además, el resultado de quasi-factorización fuerte se prueba para entropías relativas condicionadas por esperanzas generales, que se definen de la siguiente manera: Dada un álgebra de von Neumann \mathcal{M} y una subálgebra $\mathcal{N} \subset \mathcal{M}$, sea σ un estado en \mathcal{M} y $\mathcal{E}_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ la única esperanza condicionada con respecto a σ . Entonces, la *entropía relativa condicionada por esperanzas general* en \mathcal{N} se define para cada ρ como

$$D_{\mathcal{N}}^{\mathcal{E}}(\rho||\sigma) := D(\rho||\mathcal{E}_{\mathcal{N}}^*(\rho)).$$

Antes de enunciar el resultado principal del Capítulo 8, vamos a presentar dos condiciones de agrupamiento de correlaciones que constituirán suposiciones que deberemos hacer para que este resultado sea cierto.

Primero, dado \mathcal{H} un espacio de Hilbert finito dimensional, \mathcal{N}_1 y \mathcal{N}_2 dos subálgebras de von Neumann de $\mathcal{B}(\mathcal{H})$, $\mathcal{M} \subset \mathcal{N}_1 \cap \mathcal{N}_2$ otra subálgebra, y un estado σ , consideremos $\mathcal{E}_1 :$

$\mathcal{B}(\mathcal{H}) \rightarrow \mathcal{N}_i$ para $i = 1, 2$ y $\mathcal{E}_{\mathcal{M}} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{M}$ como las únicas esperanzas condicionadas en \mathcal{N}_i y \mathcal{M} con respecto a σ , respectivamente. Entonces, decimos que σ satisface \mathbb{L}_1 -agrupamiento de correlaciones condicionado con respecto a la tripla $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ si existe una constante c tal que lo siguiente se cumple para cada $X \in \mathcal{B}(\mathcal{H})$:

$$|\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(X))| \leq c \|X\|_{\mathbb{L}_1(\sigma)}^2,$$

donde la covarianza condicionada viene dada por

$$\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(X)) := \langle \mathcal{E}_1(X) - \mathcal{E}_{\mathcal{M}}(X), \mathcal{E}_2(X) - \mathcal{E}_{\mathcal{M}}(X) \rangle_{\sigma}.$$

Además, la tripla $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ satisface \mathbb{L}_1 -agrupamiento de correlaciones condicionado si todo estado $\sigma = \mathcal{E}_{\mathcal{M}}^*(\sigma)$ lo satisface con la misma constante c .

En las mismas condiciones de antes, decimos que un estado σ satisface *covarianza-entropía agrupamiento de correlaciones* con respecto a la tripla $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ si existe una constante c tal que lo siguiente se cumple para todo $X \in \mathcal{B}(\mathcal{H})$:

$$|\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(X))| \leq c D(\Gamma_{\sigma}(X) \| \Gamma_{\sigma} \circ \mathcal{E}_{\mathcal{M}}(X)),$$

donde $\Gamma_{\sigma}(X) := \sigma^{1/2} X \sigma^{1/2}$.

En ese caso, el principal resultado del Capítulo 8 es el siguiente.

Teorema 2.3.7 — QUASI-FACTORIZACIÓN FUERTE BAJO \mathbb{L}_1 -AGRUPAMIENTO DE CORRELACIONES CONDICIONADO O COVARIANZA-ENTROPÍA AGRUPAMIENTO DE CORRELACIONES, (BCR19b).

Sea \mathcal{H} un espacio de Hilbert finito dimensional y sean $\mathcal{N}_1, \mathcal{N}_2, \mathcal{M}$ subálgebras de von Neumann de $\mathcal{B}(\mathcal{H})$ tales que $\mathcal{M} \subset \mathcal{N}_1 \cap \mathcal{N}_2$. Sean $\mathcal{E}_i : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{N}_i$, para $i = 1, 2$ y $\mathcal{E}_{\mathcal{M}} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{M}$ esperanzas condicionadas con respecto a un estado σ .

Asumamos que existe una constante $0 < c < \frac{1}{2(4 + \sqrt{2})}$ tal que la tripla $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ satisface o \mathbb{L}_1 -agrupamiento de correlaciones condicionado o covarianza-entropía agrupamiento de correlaciones con correspondiente constante c . Entonces, se tiene la siguiente desigualdad para todo $\rho \in \mathcal{S}(\mathcal{H})$:

$$D_{\mathcal{M}}^{\mathcal{E}}(\rho \| \sigma) \leq \frac{1}{1 - 2(4 + \sqrt{2})c} \left(D_1^{\mathcal{E}}(\rho \| \sigma) + D_2^{\mathcal{E}}(\rho \| \sigma) \right),$$

donde $D_{\mathcal{M}}^{\mathcal{E}}(\rho \| \sigma) := D(\rho \| \mathcal{E}_{\mathcal{M}}^*(\rho))$ y $D_i^{\mathcal{E}}(\rho \| \sigma) := D(\rho \| \mathcal{E}_i^*(\rho))$ para $i = 1, 2$.

2.3.2 DESIGUALDADES LOGARÍTMICAS DE SOBOLEV

En la Parte III, nos centramos en probar positividad de constantes de log-Sobolev para ciertas dinámicas cuánticas. Nos planteamos tres problemas diferentes en tres capítulos distintos.

Primero, en el Capítulo 9 consideramos la dinámica de heat-bath con punto fijo producto tensor. Más específicamente, el Lindbladiano global en este caso se define como la suma de los locales de la siguiente forma:

$$\mathcal{L}_{\Lambda}^* := \sum_{x \in \Lambda} \mathcal{L}_x^*,$$

donde cada \mathcal{L}_x^* viene dado por $\mathcal{L}_x^* := \mathbb{E}_x^* - \mathbb{1}_{\Lambda}$ para

$$\mathbb{E}_x^*(\rho_{\Lambda}) := \sigma_{\Lambda}^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_{\Lambda}^{1/2}$$

y el punto fijo σ_{Λ} satisface

$$\sigma_{\Lambda} = \bigotimes_{x \in \Lambda} \sigma_x.$$

Puesto que σ_{Λ} es un estado producto, podemos escribir $\mathbb{E}_x^*(\rho_{\Lambda})$ como

$$\mathbb{E}_x^*(\rho_\Lambda) = \sigma_x \otimes \rho_{x^c}.$$

Por tanto, para cada $\rho_\Lambda \in \mathcal{S}_\Lambda$,

$$\mathcal{L}_\Lambda^*(\rho_\Lambda) = \sum_{x \in \Lambda} (\sigma_x \otimes \rho_{x^c} - \rho_\Lambda).$$

Entonces, para este Lindbladiano y sin ninguna suposición más, siguiendo los pasos presentados en la estrategia de la Sección 1.2, y usando en particular el Teorema 1.3.2, probamos el siguiente resultado.

Teorema 2.3.8 — CONSTANTE DE LOG-SOBOLEV PARA EL HEAT-BATH PARA PRODUCTOS TENSORES, (CLP18a).

\mathcal{L}_Λ^* definido como antes tiene una constante de log-Sobolev positiva.

A continuación, consideramos en el Capítulo 10 de nuevo la dinámica de heat-bath, pero ahora en dimensión 1, y asumimos condiciones más débiles en el punto fijo. Más específicamente, dada una cadena finita $\Lambda \subset \mathbb{Z}$ y un estado $\rho_\Lambda \in \mathcal{S}_\Lambda$, el generador de heat-bath se define como:

$$\mathcal{L}_\Lambda^*(\rho_\Lambda) = \sum_{x \in \Lambda} \left(\sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} - \rho_\Lambda \right),$$

donde el primer término de la suma de la parte derecha coincide con la aplicación de recuperación de Petz para la traza parcial en cada sitio $x \in \Lambda$, compuesta con la traza parcial en x , y σ_Λ es el estado de Gibbs de un Hamiltoniano conmutante k -local.

Necesitamos asumir que un par de condiciones de agrupamiento en el estado de Gibbs se cumplen. La primera está relacionada con el decaimiento exponencial de correlaciones en el estado de Gibbs de un Hamiltoniano conmutante y se satisface, por ejemplo, en sistemas de Gibbs clásicos. Sean $C, D \subset \Lambda$ la unión de segmentos de longitud finita no solapados de Λ . La siguiente desigualdad se cumple para constantes positivas K_1, K_2 independientes de Λ :

$$\left\| \sigma_C^{-1/2} \otimes \sigma_D^{-1/2} \sigma_{CD} \sigma_C^{-1/2} \otimes \sigma_D^{-1/2} - \mathbb{1}_{CD} \right\|_\infty \leq K_1 e^{-K_2 d(C,D)},$$

donde $d(C, D)$ es la distancia entre C y D , i.e. la distancia mínima entre dos segmentos de C y D .

La segunda condición constituye una forma más fuerte de quasi-factorización de la entropía relativa que las mencionadas anteriormente. Un ejemplo en el que se cumple es para estados de Gibbs que verifiquen $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$. En otras palabras, dado $X \subset \Lambda$, para cada $\rho_\Lambda \in \mathcal{S}_\Lambda$ se tiene la siguiente desigualdad:

$$D_X(\rho_\Lambda \| \sigma_\Lambda) \leq f_X(\sigma_\Lambda) \sum_{x \in X} D_x(\rho_\Lambda \| \sigma_\Lambda),$$

donde $1 \leq f_X(\sigma_\Lambda) < \infty$ depende solo de σ_Λ y no depende del tamaño de Λ , mientras que $D_X(\rho_\Lambda \| \sigma_\Lambda)$, resp. $D_x(\rho_\Lambda \| \sigma_\Lambda)$, es la entropía relativa condicionada en X , resp. x , de ρ_Λ y σ_Λ .

Bajo la suposición de que estas dos condiciones se cumplen, se prueba el siguiente resultado.

Teorema 2.3.9 — CONSTANTE DE LOG-SOBOLEV PARA LA DINÁMICA DE HEAT-BATH EN 1D, (Bar+19).

Sea $\Lambda \subset \mathbb{Z}$ una cadena finita. Sea $\Phi : \Lambda \rightarrow \mathcal{A}_\Lambda$ un potencial conmutante k -local, $H_\Lambda = \sum_{x \in \Lambda} \Phi(x)$ su correspondiente Hamiltoniano, y denotemos por σ_Λ al estado de Gibbs. Sea \mathcal{L}_Λ^* el generador de la dinámica de heat-bath. Si las dos anteriores condiciones se cumplen, entonces la constante de log-Sobolev de \mathcal{L}_Λ^* es estrictamente positiva e independiente de $|\Lambda|$.

Finalmente, para concluir esta parte, nos movemos en el Capítulo 11 a la dinámica de Davies. En este caso, el Lindbladiano $\mathcal{L}_\Lambda^\beta : \mathcal{A}_\Lambda \rightarrow \mathcal{A}_\Lambda$ asociado a esta dinámica para una cierta temperatura inversa finita β es de la siguiente forma:

$$\mathcal{L}_\Lambda^\beta(X) = i[H_\Lambda, X] + \sum_{k \in \Lambda} \mathcal{L}_k^\beta(X),$$

y, dado $A \subset \Lambda$, el generador local se construye restringiendo la suma de arriba a A :

$$\mathcal{L}_A^\beta(X) = i[H_A, X] + \sum_{k \in A} \mathcal{L}_k^\beta(X).$$

Ahora, definimos la *esperanza condicionada* en el álgebra \mathcal{N}_A de puntos fijos de \mathcal{L}_A^β con respecto al estado de Gibbs σ_Λ^β como sigue:

$$\mathcal{E}_A^\beta(X) := \lim_{t \rightarrow \infty} e^{t\mathcal{L}_A^\beta}(X).$$

Podemos considerar para esta esperanza condicionada la respectiva definición de la entropía relativa condicionada por esperanzas general, para la cual demostramos un resultado de quasi-factorización de la entropía relativa en el Teorema 2.3.7. Asumiendo las mismas condiciones de agrupamiento de correlaciones que se necesitaron allí, y a partir de este resultado de quasi-factorización, un argumento geométrico recursivo en la línea del mostrado para el caso clásico, y una conjetura sobre la positividad de la constante de log-Sobolev condicionada (que se nos ha comunicado que ya está probada, aunque no publicada), el siguiente resultado relativo a la desigualdad de log-Sobolev para la dinámica de Davies es cierto.

Teorema 2.3.10 — CONSTANTE DE LOG-SOBOLEV PARA LA DINÁMICA DE DAVIES, (BCR19b).

Sea $\Lambda \subset \mathbb{Z}^d$ una retícula finita y sea β una temperatura inversa finita. Consideremos $\mathcal{L}_\Lambda^{\beta*} : \mathcal{S}_\Lambda \rightarrow \mathcal{S}_\Lambda$ el Lindbladiano asociado a la dinámica de Davies y asumamos que o bien se satisface \mathbb{L}_1 -agrupamiento de correlaciones condicionado o covarianza-entropía agrupamiento de correlaciones. Entonces, si la Conjetura 11.3.1 es cierta, $\mathcal{L}_\Lambda^{\beta*}$ tiene una constante de log-Sobolev positiva que es independiente de $|\Lambda|$.

Podemos percibir que hay varias diferencias entre los resultados sobre constantes de log-Sobolev asociadas a la dinámica de heat-bath y Davies. El más notable es el hecho de que el resultado para heat-bath solo es cierto en 1D, mientras que el resultado para Davies se tiene para toda dimensión finita. Esta diferencia aparece por las diferentes geometrías empleadas en el argumento geométrico recursivo, que se diseñan de esa forma debido al hecho de que tenemos un resultado de quasi-factorización fuerte para la dinámica de Davies, mientras que solo tenemos un resultado de quasi-factorización (débil) para la dinámica de heat-bath.

Además, las condiciones que asumimos en el (conjunto de) punto fijo también difieren, aunque todas se reducen a la misma condición clásicamente, la de Dobrushin-Shlosman, debido a las condiciones DLR. Mientras que la condición de equilibración asumida para la dinámica de heat-bath parece más similar a la asumida en el artículo clásico [DPP02], las consideradas para el generador de Davies son más cercanas a las de [KB16].

2.3.3 DESIGUALDAD DE PROCESAMIENTO DE DATOS PARA LA ENTROPÍA BS

En la última parte de la tesis, cambiamos hacia un escenario más relacionado con la teoría de la información y estudiamos la desigualdad de procesamiento de datos para f -divergencias máximas.

Las f -divergencias cuánticas se emplean en teoría de la información cuántica para cuantificar la similitud de estados cuánticos. La entropía relativa es un ejemplo de las llamadas f -divergencias estándar [HM17, Section 3.2], que se definen como

$$S_f(\sigma\|\rho) := \text{tr} \left[\rho^{1/2} f(L_\sigma R_{\rho^{-1}})(\rho^{1/2}) \right]$$

para una función $f : (0, \infty) \rightarrow \mathbb{R}$ operador convexa. Aquí, L_A y R_A denotan a la multiplicación por la izquierda y por la derecha por la matriz A , respectivamente. La entropía relativa aparece cuando se toma $f(x) = x \log x$.

Sin embargo, esta no es la única forma de generalizar las f -divergencias clásicas introducidas en [AS66; Csi67]. Las f -divergencias maximales se definen como

$$\hat{S}_f(\sigma\|\rho) := \text{tr} \left[\rho f(\rho^{-1/2} \sigma \rho^{-1/2}) \right]$$

para una función operador convexa $f : (0, \infty) \rightarrow \mathbb{R}$ y fueron introducidas en [PR98]. Recientemente, se han estudiado en [Mat10], donde también se les ha dado su nombre actual (ver también [HM17, Section 3.3]). Para $f(x) = x \log x$, obtenemos la entropía relativa introducida por Belavkin y Staszewski en [BS82], que llamaremos *entropía BS* en corto:

$$\hat{S}_{\text{BS}}(\sigma\|\rho) := -\text{tr} \left[\sigma \log \left(\sigma^{-1/2} \rho \sigma^{-1/2} \right) \right].$$

Tanto las f -divergencias estándar como las maximales satisfacen una desigualdad de procesamiento de datos (DPD), es decir, dado un canal cuántico Φ , se tiene la siguiente desigualdad para cada ρ y σ :

$$S_f(\sigma\|\rho) \geq S_f(\Phi(\sigma)\|\Phi(\rho)),$$

y análogamente para las f -divergencias maximales. El estudio de condiciones de igualdad en la desigualdad previa, y más específicamente en la desigualdad de procesamiento de datos para la entropía relativa, esto es, para qué ρ, σ se cumple

$$D(\sigma\|\rho) = D(\Phi(\sigma)\|\Phi(\rho))$$

para un cierto canal cuántico Φ , ha llevado al descubrimiento de los estados de Markov cuánticos [Hay+04]. En particular, la entropía relativa se preserva si, y solo si, σ y ρ se pueden recuperar a través de la aplicación de recuperación de Petz:

$$\mathcal{R}_\Phi^\rho(X) = \rho^{1/2} \Phi^*(\Phi(\rho)^{-1/2} X \Phi(\rho)^{-1/2}) \rho^{1/2},$$

es decir, $\sigma = \mathcal{R}_\Phi^\rho(\Phi(\sigma))$ y $\rho = \mathcal{R}_\Phi^\rho(\Phi(\rho))$ [Pet03]. Esto es cierto para todas las f -divergencias estándar para las cuales f es “suficientemente complicado”. Referimos al lector a [HM17, Theorem 3.18] para encontrar una lista de condiciones equivalentes.

Para $\Phi = \mathcal{E}$ y \mathcal{E} la esperanza condicionada que preserva la traza en una subálgebra matricial unital \mathcal{N} de $\mathcal{B}(\mathcal{H})$, [CV17] muestra que la condición de igualdad es estable en el sentido de que se tiene la siguiente desigualdad:

$$D(\sigma\|\rho) - D(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \geq \left(\frac{\pi}{8}\right)^4 \|L_\rho R_{\sigma^{-1}}\|_\infty^{-2} \|\mathcal{R}_\Phi^\sigma(\rho_{\mathcal{N}}) - \rho\|_1^4. \quad (2.5)$$

Aquí hemos escrito $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$ y $\rho_{\mathcal{N}} := \mathcal{E}(\rho)$. Esto también se puede interpretar como un fortalecimiento de la desigualdad de procesamiento de datos. Trabajos posteriores han generalizado el resultado anterior a f -divergencias estándar más generales [CV18] y a la *tan-buena-como-la fidelidad* de Holevo [Wil18].

La diferencia de entropías relativas que aparece en la parte de la izquierda de la Ecuación (2.5) se ha estudiado intensivamente en el contexto de información cuántica y termodinámica cuántica [FBB18; FR18]. Más aún, siendo \mathcal{E} una traza parcial, se ha caracterizado como una entropía relativa condicionada en [CLP18a] (ver Capítulo 6). La Ecuación (2.5) es el primer fortalecimiento de la desigualdad de procesamiento de datos para la entropía relativa en función de la “distancia” entre un estado y su aplicación de recuperación de Petz, aunque ha habido muchos otros resultados con un espíritu similar en los últimos años.

En el Capítulo 12, proporcionamos resultados análogos a los de [CV17] y [CV18] para f -divergencias máximas. Para ellos, el preservar una f -divergencia máxima, es decir,

$$\hat{S}_f(\Phi(\sigma) \parallel \Phi(\rho)) = \hat{S}_f(\sigma \parallel \rho),$$

no es equivalente a que σ, ρ se puedan recuperar en el sentido de Petz, aunque lo último implica lo primero. Algunas condiciones equivalentes a la preservación de cualquier f -divergencia máxima para el caso en el que Φ sea una aplicación completamente positiva y que preserve la traza se dan en [HM17, Theorem 3.34]. En el Capítulo 12, probamos otras dos condiciones equivalentes, las cuales usaremos posteriormente para probar un fortalecimiento de la desigualdad de procesamiento de datos para f -divergencias maximales y, en particular, para la entropía BS.

Todos los sistemas cuánticos que aquí aparecen son finito dimensionales. Sean σ, ρ dos estados cuánticos definidos positivos en un álgebra matricial \mathcal{M} . Empleamos las abreviaturas: $\Gamma := \sigma^{-1/2} \rho \sigma^{-1/2}$ and $\Gamma_{\mathcal{T}} := \sigma_{\mathcal{T}}^{-1/2} \rho_{\mathcal{T}} \sigma_{\mathcal{T}}^{-1/2}$, donde \mathcal{N} es otra álgebra matricial, $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ es una aplicación completamente positiva y que preserve la traza y $\rho_{\mathcal{T}} := \mathcal{T}(\rho)$, $\sigma_{\mathcal{T}} := \mathcal{T}(\sigma)$. Nuestro primer resultado consiste en dos condiciones que son equivalentes a que se preserve la entropía BS bajo \mathcal{T} . Se sigue del Teorema 12.2.2 junto con la Proposición 12.2.5 y el Teorema 12.5.1.

Teorema 2.3.11 — CONDICIONES DE IGUALDAD EN LA DPD PARA LA ENTROPÍA BS, (BC19b).

Sean \mathcal{M} y \mathcal{N} dos álgebras matriciales y sean $\sigma > 0, \rho > 0$ dos estados cuánticos en \mathcal{M} . Sea $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ una aplicación completamente positiva y que preserve la traza y sea V la isometría asociada a la dilatación de Stinespring (Teorema 4.4.9) de \mathcal{T} . Entonces, las siguientes condiciones son equivalentes:

1. $\hat{S}_{\text{BS}}(\sigma \parallel \rho) = \hat{S}_{\text{BS}}(\sigma_{\mathcal{T}} \parallel \rho_{\mathcal{T}})$
2. $\sigma^{-1} \rho = \mathcal{T}^*(\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}})$
3. $V \sigma^{1/2} V^* \left(\sigma_{\mathcal{T}}^{-1/2} \Gamma_{\mathcal{T}}^{1/2} \sigma_{\mathcal{T}}^{1/2} \otimes I \right) = V \Gamma^{1/2} \sigma^{1/2} V^*.$

El anterior teorema viene motivado por el tratamiento realizado en [Pet03] sobre la entropía relativa y sigue las mismas líneas. Este resultado permite un fortalecimiento de la desigualdad de procesamiento de datos para la entropía BS, a partir del trabajo de [CV17] para esperanzas condicionadas, y posteriormente refinando el resultado a canales cuánticos generales usando el teorema de dilatación de Stinespring:

Teorema 2.3.12 — DPD FORTALECIDA PARA LA ENTROPÍA BS, (BC19b).

Sean \mathcal{M} y \mathcal{N} dos álgebras matriciales y sea $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ una aplicación completamente positiva y que preserve la traza. Sean σ, ρ dos estados cuánticos en \mathcal{M} con el mismo soporte. Entonces, se tiene

$$\hat{S}_{\text{BS}}(\sigma \parallel \rho) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{T}} \parallel \rho_{\mathcal{T}}) \geq \left(\frac{\pi}{8} \right)^4 \|\Gamma\|_{\infty}^{-4} \|\sigma_{\mathcal{T}}^{-1}\|_{\infty}^{-2} \|\sigma \mathcal{T}^*(\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}}) - \rho\|_2^4. \quad (2.6)$$

El Teorema 2.3.11 muestra que la parte de la derecha de la Ecuación (2.6) juega el mismo papel que la distancia tracial entre ρ y el estado obtenido de la aplicación de recuperación en

la Ecuación (2.5). El resultado para esperanzas condicionadas aparece en el Corolario 12.3.5 en el texto principal y se sigue de la mejora en la acotación inferior del Teorema 12.3.3. Estos resultados son posteriormente refinados a canales cuánticos generales en el Teorema 12.5.1.

En el resto del trabajo, extendemos los resultados de la entropía BS a f -divergencias máximas más generales. Este enfoque es similar al trabajo realizado en [CV18]. Consideramos funciones operador convexas $f : (0, \infty) \rightarrow \mathbb{R}$ cuya transpuesta $\tilde{f}(x) := xf(1/x)$ es operador monótona decreciente. Además, asumimos que la medida $\mu_{-\tilde{f}}$ de $-\tilde{f}$ es absolutamente continua con respecto a la medida de Lebesgue y que existen $C > 0$, $\alpha \geq 0$ tales que, para cada $T \geq 1$, la derivada de Radon-Nikodým está inferiormente acotada por

$$\frac{d\mu_{-\tilde{f}}(t)}{dt} \geq (CT^{2\alpha})^{-1}$$

casi por doquier (con respecto a la medida de Lebesgue) para todo $t \in [1/T, T]$. Más aún, asumimos que nuestros estados $\sigma > 0$, $\rho > 0$ no están muy lejos de satisfacer la desigualdad de procesamiento de datos con respecto a \mathcal{E} , es decir,

$$\left(\frac{(2\alpha + 1)\sqrt{C}}{4} \frac{(\hat{S}_f(\sigma\|\rho) - \hat{S}_f(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}))^{1/2}}{1 + \|\Gamma\|_{\infty}} \right)^{\frac{1}{1+\alpha}} \leq 1. \quad (2.7)$$

Teorema 2.3.13 — ESTABILIDAD PARA LA DPD PARA f -DIVERGENCIAS MÁXIMALES, (BC19b).

Sean \mathcal{M} y \mathcal{N} dos álgebras matriciales y sea $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ una aplicación completamente positiva y que preserva la traza. Sean σ, ρ dos estados cuánticos en \mathcal{M} con el mismo soporte y sea $f : (0, \infty) \rightarrow \mathbb{R}$ una función operador convexa con transpuesta \tilde{f} . Asumamos que \tilde{f} es operador monótona decreciente y tal que la medida $\mu_{-\tilde{f}}$ que aparece en Teorema 4.4.2 es absolutamente continua con respecto a la medida de Lebesgue. Además, asumamos que para cada $T \geq 1$, existen constantes $\alpha \geq 0$, $C > 0$ satisfaciendo $d\mu_{-\tilde{f}}(t)/dt \geq (CT^{2\alpha})^{-1}$ para todo $t \in [1/T, T]$ y tales que la Ecuación (1.7) se cumple. Entonces, existe una constante $L_{\alpha} > 0$ tal que

$$\begin{aligned} & \hat{S}_f(\sigma\|\rho) - \hat{S}_f(\sigma_{\mathcal{T}}\|\rho_{\mathcal{T}}) \\ & \geq \frac{L_{\alpha}}{C} (1 + \|\Gamma\|_{\infty})^{-(4\alpha+2)} \|\Gamma\|_{\infty}^{-(2\alpha+2)} \|\sigma_{\mathcal{T}}^{-1}\|_{\infty}^{-(2\alpha+2)} \|\rho - \sigma_{\mathcal{T}}^* (\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}})\|_2^{4(\alpha+1)}. \end{aligned}$$

Para esperanzas condicionadas, el anterior resultado aparece como Corolario 12.4.2 en el texto principal y se sigue de la mejora en la acotación inferior que aparece en el Teorema 12.4.1. La extensión a canales cuánticos generales aparece en el Teorema 12.5.3.

2.4 ORGANIZACIÓN DE LA TESIS

Los contenidos de la tesis están organizados como sigue. En el Capítulo 1, damos una introducción a los problemas trabajados en esta tesis, los resultados que se han probado e introducimos algo de notación (con versión traducida al castellano en el Capítulo 2). Posteriormente, en el Capítulo 3, introducimos sistemas de espines clásicos, comentamos el problema análogo clásico de probar positividad de constantes de log-Sobolev y desarrollamos la estrategia seguida en este resultado que se basa en resultados de quasi-factorización de la entropía. Concluimos la parte introductoria de la tesis revisando algunas nociones y propiedades preliminares que serán necesarias para entender el resto del texto en el Capítulo 4.

En la Parte II, nos centramos en resultados de quasi-factorización de la entropía relativa. Primero, en el Capítulo 5, presentamos una extensión cuantitativa de la propiedad de superaditividad de la entropía relativa para estados generales. Tras introducir y caracterizar varios

conceptos de entropía relativa condicionada en el Capítulo 6, demostramos algunos resultados de quasi-factorización de la entropía relativa para diferentes entropías relativas condicionadas en el Capítulo 7. Posteriormente, presentamos algunas versiones más fuertes de estos resultados de quasi-factorización en el Capítulo 8.

Cambiamos al estudio sobre desigualdades logarítmicas de Sobolev en la Parte III. Este estudio comienza con el caso particular de un producto tensor como punto fijo de la evolución correspondiente a la dinámica de heat-bath en el Capítulo 9, para el que demostramos que la constante de log-Sobolev está siempre inferiormente acotada por $1/2$. Después, consideramos de nuevo la dinámica de heat-bath, pero ahora asumiendo condiciones más débiles en el punto fijo de la evolución y demostramos en el Capítulo 10 que, si de hecho corresponde al estado de Gibbs de un Hamiltoniano conmutante local, bajo dos condiciones de agrupamiento de correlaciones en este estado, la constante de log-Sobolev asociada a sistemas 1D es positiva. Para concluir esta parte, nos trasladamos a la dinámica de Davies en el Capítulo 11, para la que nos planteamos el problema de probar positividad de las constantes de log-Sobolev bajo ciertas condiciones de agrupamiento de correlaciones, via los resultados de quasi-factorización fuertes mencionados anteriormente.

Finalmente, en la Parte IV, y más específicamente en el Capítulo 12, consideramos el problema de fortalecer la desigualdad de procesamiento de datos asociada a la entropía BS. Primero, proporcionamos dos nuevas condiciones que son equivalentes a tener igualdad en la desigualdad de procesamiento de datos asociada a la entropía BS, lo cual permite definir una condición de recuperación BS. Posteriormente, usamos estas condiciones para proporcionar una versión fortalecida de la desigualdad de procesamiento de datos para la entropía BS y, en mayor generalidad, para una gran clase de f -divergencias maximales.

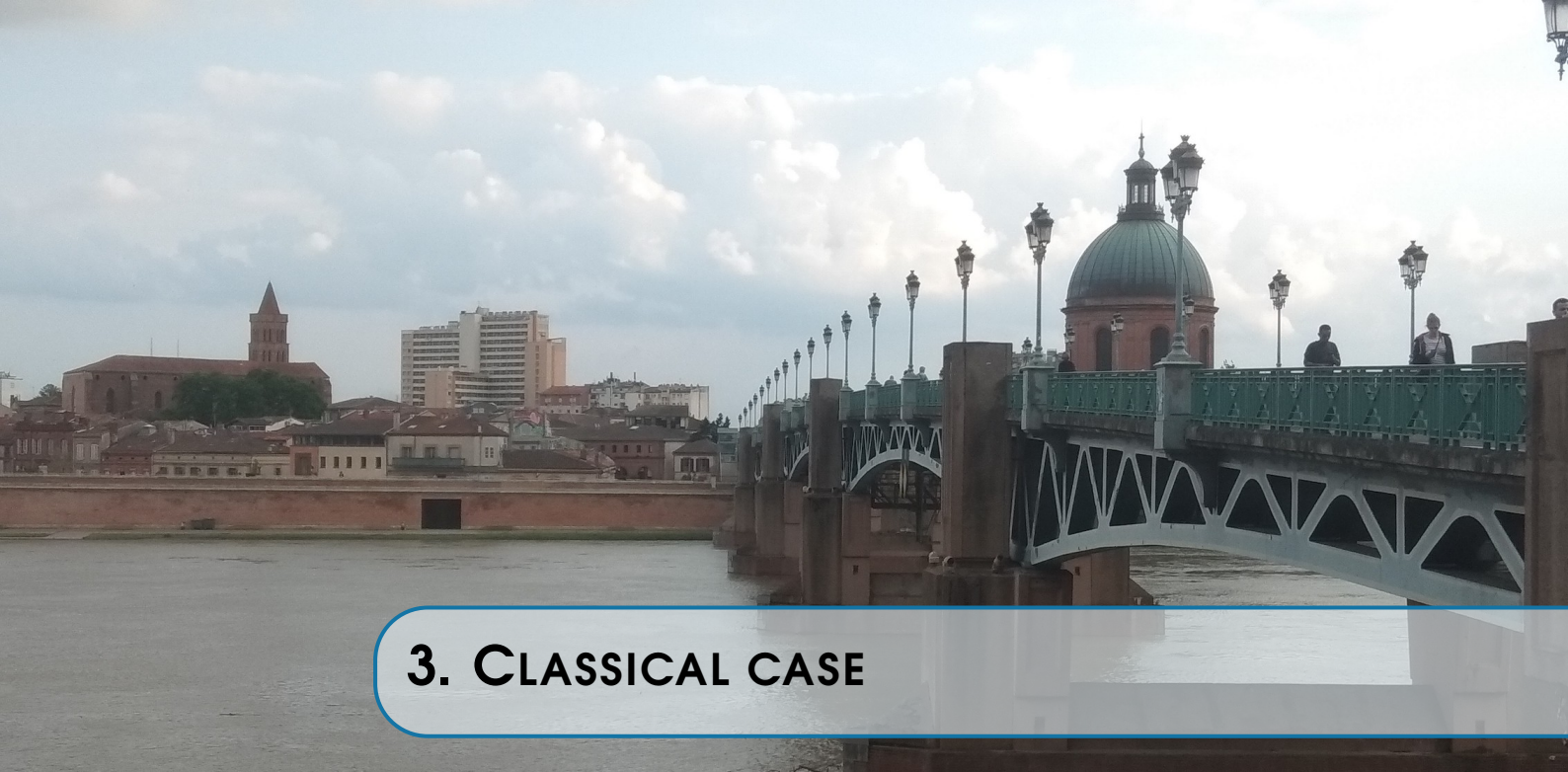
Para concluir, los principales resultados de esta tesis se han comunicado en las siguientes publicaciones científicas:

- (CLP18b) Á. Capel, A. Lucia and D. Pérez-García, *Superadditivity of Quantum Relative Entropy for General States*, *IEEE Trans. Inf. Theory*, 64 (7) (2018), 4758-4765, DOI: [10.1109/TIT.2017.2772800](https://doi.org/10.1109/TIT.2017.2772800), arXiv: [1705.03521](https://arxiv.org/abs/1705.03521), (Capítulo 5).
- (CLP18a) Á. Capel, A. Lucia and D. Pérez-García, *Quantum Conditional Relative Entropy and Quasi-Factorization of the Relative Entropy*, *J. Phys. A: Math. Theor.*, 51 (2018), 484001, DOI: [10.1088/1751-8121/aae4cf](https://doi.org/10.1088/1751-8121/aae4cf), arXiv: [1804.09525](https://arxiv.org/abs/1804.09525), (Capítulos 6, 7 y 9).
- (BC19b) A. Bluhm and Á. Capel, *A strengthened data processing inequality for the Belavkin-Staszewski relative entropy*, *Rev. Math. Phys.*, to appear (2019), DOI: [10.1142/S0129055X20500051](https://doi.org/10.1142/S0129055X20500051), arXiv: [1904.10768](https://arxiv.org/abs/1904.10768), (Capítulo 12).
- (Bar+19) I. Bardet, Á. Capel, A. Lucia, D. Pérez-García and C. Rouzé, *On the modified logarithmic Sobolev inequality for the heat-bath dynamics for 1D systems*, preprint (2019), arXiv: [1908.09004](https://arxiv.org/abs/1908.09004), (Capítulo 10 y Sección 8.2).
- (BCR19b) I. Bardet, Á. Capel and C. Rouzé, *Positivity of the modified logarithmic Sobolev constant for quantum Davies semigroups: the commuting case*, in preparation (2019), (Capítulos 8 y 11).

Otro resultado en una línea de investigación diferente que la candidata ha obtenido durante su doctorado, y que no se incluye en el núcleo principal de la tesis para homogeneizar lo máximo

posible, pero se mencionará brevemente en el Apéndice 12.5, está basado en el siguiente artículo:

- (CMM17) Á. Capel, M. Martín and J. Merí, *Numerical radius attaining compact linear operators*, *J. Math. Anal. Appl.*, 445 (2017), 1258-1266,
DOI: [10.1016/j.jmaa.2016.02.074](https://doi.org/10.1016/j.jmaa.2016.02.074), arXiv: [1602.07084](https://arxiv.org/abs/1602.07084).



3. CLASSICAL CASE

In this chapter, we present a brief review on classical spin systems and the analogous result in this setting to the main results in the quantum setting shown in this thesis. Namely, we start introducing some notation and basic concepts related to classical lattice spin models, to subsequently review Gibbs measures (the classical analogue of quantum Gibbs states) and briefly study those measures in the most famous model of lattice spin systems, the Ising model.

Afterwards, we introduce the dynamics associated to Markov generators and some constants that can be used to study the ergodicity of the Markov semigroup associated to that generator. Finally, after reviewing weak and strong mixing conditions to be assumed on the Gibbs measure, we sketch the proof of a classical result in which an entropy constant (analogous to our quantum log-Sobolev constant) is shown to be positive, whose strategy constitutes the basis to construct our strategy to prove positivity of quantum log-Sobolev constants (see Section 1.2).

3.1 NOTATION AND BASIC CONCEPTS

Let us start by introducing some concepts and notation concerning lattice spin models. One of the main references for such models is [Mar99] and we will mainly use here the notation presented in [DPP02].

Definition 3.1.1 — LATTICE AND SITES.

We call the set \mathbb{Z}^d a *d-dimensional lattice*, where the elements $x \in \mathbb{Z}^d$ are called *sites*, and we equip \mathbb{Z}^d with the norm given by

$$|x| = \max_{i \in \{1, \dots, d\}} |x_i|$$

for every $x = \{x_1, \dots, x_d\}$.

We denote the associated distance function by $d(\cdot, \cdot)$, which is given for $X, Y \subset \mathbb{Z}^d$ by

$$d(X, Y) := \min\{|x - y| : x \in X, y \in Y\},$$

This is a picture of Toulouse (France) during the “Workshop on quantum functional inequalities” that took place there in June 2018.

although we will more often use the following distance for every two sites of the lattice

$$d_2(x, y) = \left(\sum_{i=1}^d |x_i - y_i|^2 \right)^{1/2}.$$

Definition 3.1.2 — RECTANGLE.

Let $x \in \mathbb{Z}^d$ be a site and $l_1, \dots, l_d \in \mathbb{N}$. We can define the following *rectangle*:

$$R(x; l_1, \dots, l_d) := x + ([1, l_1] \times \dots \times [1, l_d]) \cap \mathbb{Z}^d. \quad (3.1)$$

Given a rectangle of this form, we define its *size* by $\max\{l_k : k = 1, \dots, d\}$, and we say that the rectangle is *fat* if

$$\min\{l_k : k = 1, \dots, d\} \geq \frac{1}{10} \max\{l_k : k = 1, \dots, d\}. \quad (3.2)$$

One particular case of rectangle appears when the size of all the sides coincide. In this case, the rectangle is called a *cube* and denoted by Q_L , where $L = l_i$ for every $i = 1, \dots, d$.

Let us denote by \mathcal{R}_L the class of all fat rectangles in \mathbb{Z}^d of size at most $L \in \mathbb{N}$ and $\mathcal{R} = \bigcup_{L \geq 1} \mathcal{R}_L$.

Note that Q_L stands for the cube of size L starting at the origin. For a site $x \in \mathbb{Z}^d$, we denote by $Q_L(x)$ the cube given by $Q_L + \{x\}$. We further denote by B_L the ball of radius L centered at the origin, that is $B_L = Q_{2L+1}(\{-L, \dots, -L\})$.

Given a finite subset Λ of \mathbb{Z}^d , which we denote by $\Lambda \subset \subset \mathbb{Z}^d$, and whose cardinality is written as $|\Lambda|$, we say that it is a *multiple* of Q_L if there exists $y \in Q_L$ such that Λ is the union of a finite number of cubes of the form $Q_L(x_i + y)$, for $x_i \in L\mathbb{Z}^d$.

For $\Lambda \subset \subset \mathbb{Z}^d$, we define its *r-boundary* by $\partial_r^+ \Lambda := \{x \in \Lambda^c : d(x, \Lambda) \leq r\}$, where $d(x, \Lambda) := \inf_{y \in \Lambda} d(x, y)$. Note that we are only considering in this definition the outer boundary. Moreover, we say that a region Λ is connected if for every $x, y \in \Lambda$ there exist $\{z^1, \dots, z^m\} \subset \Lambda$ such that $x = z^1$, $y = z^m$ and $d(z^i, z^{i+1}) = 1$ for every i .

Definition 3.1.3 — CONFIGURATION SPACE.

The *configuration space* is defined as $S^{\mathbb{Z}^d}$, for S a certain set called the *single spin space*, and denoted by Ω . We will only consider the case for $S = \{-1, 1\}$ or $S = \mathbb{N}$ and, for every $V \subset \mathbb{Z}^d$, we will write $\Omega_V := S^V$.

The space S is endowed with the discrete topology. Thus, in Ω we consider the corresponding product topology, i.e., the Borel σ -algebra \mathcal{F} generated by the open sets of the product topology.

For a configuration $\sigma \in \Omega$, we denote by σ_x its value at $x \in \mathbb{Z}^d$, and given a subset $\Lambda \subset \mathbb{Z}^d$, we denote by σ_Λ the natural projection over Ω_Λ , the reduced configuration space. Note that it will have associated a σ -algebra \mathcal{F}_Λ which is generated by $\{\sigma_x : x \in \Lambda\}$.

Moreover, if we consider two configurations σ, η and two disjoint sets $X, Y \subset \mathbb{Z}^d$, then we write $\sigma_X \eta_Y$ for the configuration on $X \cup Y$ which is equal to σ on X and η on Y . In general, if X and Y are not disjoint, $\sigma \in \Omega_X$ and $\eta \in \Omega_Y$, we define $\sigma \eta \in \Omega_{X \Delta Y}$, where $X \Delta Y := (X \setminus Y) \cup (Y \setminus X)$ denotes the *symmetric difference* between X and Y , given by

$$(\sigma \eta)_x := \begin{cases} \sigma_x, & x \in X \setminus Y, \\ \eta_x, & x \in Y \setminus X. \end{cases}$$

If f is a function on Ω , we denote by Λ_f the smallest subset of \mathbb{Z}^d such that $f(\sigma)$ only depends on σ_{Λ_f} . We denote the supremum norm of f by

$$\|f\|_u := \sup_{\omega \in \Omega} |f(\omega)|.$$

When the single spin space considered is $S = \{-1, 1\}$, the *gradient* of a function f is defined as

$$(\nabla_x f)(\sigma) := f(\sigma^x) - f(\sigma),$$

where σ^x stands for the configuration obtained from σ by flipping the spin at site $x \in \mathbb{Z}^d$. In general, for $\Lambda \subset \mathbb{Z}^d$, we define the following generalized gradient as

$$|\nabla_\Lambda f|^2 := \sum_{x \in \Lambda} (\nabla_x f)^2.$$

On the other side, when $S = \mathbb{N}$, we define the following two gradients, which will be of use for the definition of the Markov generator in subsequent sections:

$$(\nabla_x^- f)(\sigma) := \chi_{\{\sigma_x > 0\}} [f(\sigma - \delta^x) - f(\sigma)],$$

$$(\nabla_x^+ f)(\sigma) := f(\sigma + \delta^x) - f(\sigma),$$

for $x \in \mathbb{Z}^d$, and χ_X the characteristic function of the set X , and where the configuration $\delta \in \Omega$ is given by

$$(\delta^x)_y := \begin{cases} 1, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

3.2 GIBBS MEASURES

Let us start this section by introducing the appropriate potential from which we will define the Hamiltonian later.

Definition 3.2.1 — BOUNDED, FINITE RANGE, TRANSLATIONAL-INVARIANT POTENTIAL.

We define a *bounded, finite range, translational-invariant potential*, with range $r > 0$, as a collection $\Phi := \{\Phi_\Lambda : \Lambda \subset \mathbb{Z}^d\}$ such that, for every $\Lambda \subset \mathbb{Z}^d$, Φ_Λ is a function $\Phi_\Lambda : \Omega_\Lambda \rightarrow \mathbb{R}$ verifying:

1. $\Phi_\Lambda = \Phi_{\Lambda + \{x\}}$ for all $x \in \mathbb{Z}^d$.
2. $\Phi_\Lambda = 0$ if $\text{diam}(\Lambda) > r$.
3. $\|\Phi\| := \sup_{x \in \mathbb{Z}^d} \sum_{\Lambda \ni x} |\Phi_\Lambda| < \infty$.

From a potential verifying the above properties, given $V \subset \mathbb{Z}^d$, we define the *Hamiltonian* $H_{V,\Phi} : \Omega \rightarrow \mathbb{R}$ by

$$H_{V,\Phi}(\sigma) := - \sum_{\Lambda : \Lambda \cap V \neq \emptyset} \Phi_\Lambda(\sigma_\Lambda).$$

We will drop the subindex “ Φ ” when the potential is clear. Note that

$$\|H_V\|_u \leq |V| \|\Phi\|,$$

where $|V|$ stands for the cardinality of V . Moreover, for $\tau \in \Omega$, we write $H_V^\tau := H_V(\sigma_V \tau_{V^c})$, where τ is called the *boundary condition*.

Definition 3.2.2 — CONDITIONAL GIBBS MEASURE.

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice and consider $\tau \in \Omega$. We define the (finite) conditional *Gibbs measure* on (Ω, \mathcal{F}) by

$$\mu_\Lambda^\tau(\sigma) := (Z_\Lambda^\tau)^{-1} \exp[-H_\Lambda^\tau(\sigma)] \prod_{x \in \Lambda} \rho(\sigma_x), \quad (3.3)$$

where Z_Λ^τ is the *normalization factor* (also called frequently *partition function*) and $\rho(\cdot)$ is a certain *reference measure* on \mathbb{N} .

The only reference measure that we will consider later is the *Poisson measure*, which is given by

$$\rho(n) = e^{-\lambda} \frac{\lambda^n}{n!}.$$

R Remark 3.2.3

Note that, although in the expression of the Hamiltonian and the Gibbs measure the inverse temperature factor β does not appear explicitly, it is absorbed in the definition of the potential Φ . Throughout the whole manuscript we will omit the dependence of β to avoid confusion, except in Chapter 11, where it will appear explicitly since we will compare these quantities at different temperatures.

Analogously to what we mentioned for the Hamiltonian, we also drop the dependence on the potential of the Gibbs measure when it is unnecessary to remark it, as we did above. Given a measurable function f on Ω , $\mu_\Lambda^\tau(f)$ denotes the expectation of f with respect to the measure μ_Λ^τ , that is its average. Moreover, when the superscript is omitted, we denote by $\mu_\Lambda(f)$ the function $\sigma \mapsto \mu_\Lambda^\sigma(f)$. We further write $\mu_\Lambda(X) := \mu_\Lambda(\chi_X)$ for every $X \in \mathcal{F}$, where χ_X is the characteristic function on X .

The set of measures introduced in Equation (3.3) satisfies the *DLR compatibility conditions* [Dob68] [LR69]

$$\mu_\Lambda^\tau(\mu_V(X)) = \mu_\Lambda^\tau(X), \quad \forall X \in \mathcal{F}, \quad \forall V \subset \Lambda \subset \mathbb{Z}^d. \quad (3.4)$$

This motivates the definition of a family of Gibbs measures as probability measures satisfying the DLR conditions.

Definition 3.2.4 — GIBBS MEASURE.

A probability measure μ on (Ω, \mathcal{F}) is called a *Gibbs measure* for the potential Φ if

$$\mu(\mu_V(X)) = \mu(X) \quad \forall X \in \mathcal{F}, \quad \forall V \subset \mathbb{Z}^d.$$

R Remark 3.2.5

Note that $\mu_\Lambda(f)$ is measurable with respect to \mathcal{F}_{Λ^c} . Since for every g also measurable w.r.t. \mathcal{F}_{Λ^c} we have $\mu_\Lambda^\sigma(fg) = g(\sigma)\mu_\Lambda^\sigma(f)$, we can understand the DLR conditions, i.e., Equation (3.4) as an equivalent way to say that $\mu_\Lambda(f)$ is a version of the conditional expectation $\mu_\Lambda^\tau(f|\mathcal{F}_{V^c})$.

The measure introduced in Equation (3.3) is clearly a Gibbs measure according to the condition presented in the definition above, and that is the reason for the name given to the former. However, in general it is not the only Gibbs measure for a certain potential. Indeed, the set of all Gibbs measures associated to a certain potential Φ will be denoted by \mathcal{G} , and it can be

proven that it is nonempty, convex and compact. We say that the discrete spin system described by Φ exhibits a *phase transition* if \mathcal{G} has more than one element.

Next, let us introduce some notions that will appear in the results of the rest of the chapter. Given measurable functions f and g , we define their *covariance* w.r.t. μ_Λ^τ by

$$\mu_\Lambda^\tau(f, g) := \mu_\Lambda^\tau(fg) - \mu_\Lambda^\tau(f)\mu_\Lambda^\tau(g).$$

When f and g coincide, the covariance just reduces to the *variance*. Given $V \subset \Lambda$, we denote by $\mu_{\Lambda, V}$ the *marginal* of μ_Λ in Ω_V , i.e., $\mu_\Lambda(f) = \mu_{\Lambda, V}(f)$ for any f measurable w.r.t. \mathcal{F}_V .

Given a probability space $(\Omega, \mathcal{F}, \mu)$, we define, for every $f > 0$, the *entropy* of f by

$$\text{Ent}_\mu(f) := \mu(f \log f) - \mu(f) \log \mu(f),$$

for $f \log f \in \mathbb{L}_1(\mu)$, and $\text{Ent}_\mu(f) = +\infty$ otherwise. It is known that $\text{Ent}_\mu(f) = 0$ if, and only if, f is constant μ -a.s.

Consider now two probability measures μ and ν on $(\Omega_\Lambda, \mathcal{F}_\Lambda)$ such that ν is absolutely continuous with respect to μ . We define the *relative entropy* of ν with respect to μ by

$$H(\nu|\mu) := \mu(f \log f),$$

where f is the Radon-Nikodym derivative of ν with respect to μ .

Note that the relation between the notions of entropy and relative entropy is given by

$$H(\nu|\mu) = \text{Ent}_\mu\left(\frac{d\nu}{d\mu}\right).$$

Moreover, we define the *total variation distance* between μ and ν by

$$\|\mu - \nu\|_{\text{TV}} := \frac{1}{2} \sum_{\sigma \in \Omega_\Lambda} |\mu(\sigma) - \nu(\sigma)| = \max_{X \subset \Omega_\Lambda} |\mu(X) - \nu(X)|.$$

To conclude this section, we will present some information on the most famous example of lattice spin system, the *Ising model*.

3.2.1 THE ISING MODEL

Let us study now the Ising model. For this model, the potential Φ introduced above takes the following values:

$$\Phi_\Lambda = \beta \begin{cases} 1 & \text{if } \Lambda = \{x, y\} \text{ with } d_2(x, y) \leq 1, \\ h & \text{if } \Lambda = x, \\ 0 & \text{otherwise,} \end{cases}$$

where β is the inverse temperature and h is the external magnetic field.

For this model, in dimension larger than 1, there exists a finite value β_c , the *critical inverse temperature*, such that there exists a unique Gibbs measure for any $\beta < \beta_c$ or $h \neq 0$ [Pfi91]. However, if $h = 0$ and $\beta > \beta_c$, there is a *phase transition*.

In the latter case, there exist two Gibbs measures, which are usually denoted by μ_\pm^β , which can be obtained in the thermodynamic limit, as $L \rightarrow \infty$ in the finite volume Gibbs measures $\mu_{B_L}^{\beta, \pm}$.

3.3 THE DYNAMICS

The stochastic dynamics that we want to study in this chapter are determined by a Markov generator \mathcal{L}_Λ^τ , which is defined for a lattice $\Lambda \subset \mathbb{Z}^d$ and a boundary condition $\tau \in \Omega$ by

$$(\mathcal{L}_\Lambda^\tau f)(\eta) := \sum_{x \in \Lambda, * \in \{-, +\}} c_\Lambda^\tau(x, \eta, *) (\nabla_x^* f)(\eta),$$

for every η on Ω , where $c_\Lambda^\tau(\cdot, \cdot, \cdot)$ are the *rates*, which satisfy the *detailed balance condition* with respect to μ_Λ^τ :

$$c_\Lambda^\tau(x, \eta, \pm) \mu_\Lambda^\tau(\eta) = c_\Lambda^\tau(x, \eta \pm \delta_x, \mp) \mu_\Lambda^\tau(\eta \pm \delta_x),$$

for every $\eta \in \Omega_\Lambda$ and $x \in \Lambda$, which means that \mathcal{L}_Λ^τ is self-adjoint in $\mathbb{L}_2(\mu_\Lambda^\tau)$. We need to further assume that there exists a positive constant C , which might only depend on β and the potential, and such that the following holds:

$$C^{-1} \bar{c}(x, \eta, \pm) \leq c_\Lambda^\tau(x, \eta, \pm) \leq C \bar{c}(x, \eta, \pm)$$

for every $\eta, \tau \in \Omega_\Lambda$ and $x \in \mathbb{Z}^d$, where the $\bar{c}(x, \eta, \pm)$ are the rates for a system with the same reference measure and with no interaction (which will appear in Proposition 3.5.3, for instance). From these \bar{c} rates, one example of rates satisfying the above conditions can be defined by:

$$c_\Lambda^\tau(x, \eta, \pm) := \bar{c}(x, \eta, \pm) \exp \left[-\frac{\beta}{2} \nabla_x^\pm H_\Lambda^\tau(\eta) \right].$$

Moreover, the *Markov semigroup* generated by \mathcal{L}_Λ^τ in $\mathbb{L}_2(\mu_\Lambda^\tau)$ is denoted by $\{e^{t\mathcal{L}_\Lambda^\tau}\}_{t \geq 0}$.

Now, let us introduce some notions of relevance for the rest of the chapter and which will be extended to quantum versions of them later in this text. First, we introduce the *Dirichlet form*.

Definition 3.3.1 — DIRICHLET FORM.

Let \mathcal{L}_Λ^τ be a Markov generator for a lattice $\Lambda \subset \mathbb{Z}^d$ and a boundary condition $\tau \in \Omega$. The *Dirichlet form* associated to \mathcal{L}_Λ^τ is defined by:

$$\mathcal{E}_\Lambda^\tau(f, g) := -\mu_\Lambda^\tau(f \mathcal{L}_\Lambda^\tau g) = \sum_{x \in \Lambda} \mu_\Lambda^\tau(c_\Lambda^\tau(x, \cdot, +) (\nabla_x^+ f)(\cdot) (\nabla_x^+ g)(\cdot)).$$

Next, taking into account the definition just presented for the Dirichlet form associated to a Markov generator, we can define the *spectral gap* as the optimal constant for the Poincaré inequality.

Definition 3.3.2 — SPECTRAL GAP.

Let \mathcal{L}_Λ^τ be a Markov generator for a lattice $\Lambda \subset \mathbb{Z}^d$ and a boundary condition $\tau \in \Omega$. The *spectral gap* associated to \mathcal{L}_Λ^τ is defined by:

$$\text{gap}(\mathcal{L}_\Lambda^\tau) := \inf \left\{ \frac{\mathcal{E}_\Lambda^\tau(f, f)}{\mu_\Lambda^\tau(f, f)}, f \in \mathbb{L}_2(\mu_\Lambda^\tau), \mu_\Lambda^\tau(f, f) \neq 0 \right\}.$$

From this notion of spectral gap, one can derive the following inequality, which concerns the convergence to equilibrium of the semigroup $\{e^{t\mathcal{L}_\Lambda^\tau}\}_{t \geq 0}$:

$$\left\| e^{t\mathcal{L}_\Lambda^\tau} f - \mu_\Lambda^\tau(f) \right\|_{\mathbb{L}_2(\mu_\Lambda^\tau)} \leq \|f\|_{\mathbb{L}_2(\mu_\Lambda^\tau)} e^{-\text{gap}(\mathcal{L}_\Lambda^\tau)t/2}.$$

Analogously to what we have done above for the spectral gap, we can now introduce the *entropy constant*.

Definition 3.3.3 — ENTROPY CONSTANT.

Let \mathcal{L}_Λ^τ be a Markov generator for a lattice $\Lambda \subset \mathbb{Z}^d$ and a boundary condition $\tau \in \Omega$. The *entropy constant* associated to \mathcal{L}_Λ^τ is defined by:

$$s(\mathcal{L}_\Lambda^\tau) := \inf \left\{ \frac{\mathcal{E}_\Lambda^\tau(f, \log f)}{\text{Ent}_{\mu_\Lambda^\tau}(f)}, f \geq 0, f \log f \in \mathbb{L}_1(\mu_\Lambda^\tau), \text{Ent}_{\mu_\Lambda^\tau}(f) \neq 0 \right\}.$$

This quantity is the optimal constant associated to the entropy inequality

$$\text{Ent}_{\mu_\Lambda^\tau}(f) \leq (s(\mathcal{L}_\Lambda^\tau))^{-1} \mathcal{E}_\Lambda^\tau(f, \log f),$$

and along with Csiszar's inequality for measures,

$$\|\mu - \nu\|_{\text{TV}} \leq \sqrt{\frac{1}{2} H(\nu | \mu)},$$

allows to prove the following inequality

$$\left\| \nu e^{t\mathcal{L}_\Lambda^\tau} - \mu_\Lambda^\tau \right\|_{\text{TV}} \leq \sqrt{\frac{1}{2} H(\nu | \mu_\Lambda^\tau) e^{-s(\mathcal{L}_\Lambda^\tau)t/2}}.$$

Note that a completely analogous inequality in the quantum setting is the one that allows to obtain conditions for rapid mixing from the existence of positive quantum log-Sobolev constants. Let us now introduce another constant which is the optimal constant of a certain functional inequality, the *logarithmic Sobolev constant*, and which can be also used to obtain bounds for the convergence to equilibrium of the previous semigroup.

Definition 3.3.4 — LOGARITHMIC SOBOLEV CONSTANT.

Let \mathcal{L}_Λ^τ be a Markov generator for a lattice $\Lambda \subset \mathbb{Z}^d$ and a boundary condition $\tau \in \Omega$. The *logarithmic Sobolev constant* associated to \mathcal{L}_Λ^τ is defined by:

$$S(\mathcal{L}_\Lambda^\tau) := \inf \left\{ \frac{\mathcal{E}_\Lambda^\tau(\sqrt{f}, \sqrt{f})}{\text{Ent}_{\mu_\Lambda^\tau}(f)}, f \geq 0, f \log f \in \mathbb{L}_1(\mu_\Lambda^\tau), \text{Ent}_{\mu_\Lambda^\tau}(f) \neq 0 \right\}.$$

R Remark 3.3.5

It is important to remark that the notion of classical logarithmic Sobolev constant and the quantum one presented in this text do not agree. The notion we have just introduced in the classical setting, coincides with the quantum so-called 2-logarithmic Sobolev constant, whereas the quantum logarithmic Sobolev constant is the quantum extension of the entropy constant introduced classically above.

To conclude this section, let us compare these three constants for the same Markov generator. First, it was proven in [DS96] that

$$\mu_\Lambda^\tau(f \mathcal{L}_\Lambda^\tau \log f) \leq 4\mu_\Lambda^\tau(\sqrt{f} \mathcal{L}_\Lambda^\tau \sqrt{f}),$$

and thus,

$$4s(\mathcal{L}_\Lambda^\tau) \geq S(\mathcal{L}_\Lambda^\tau),$$

which implies the fact that if a generator has a positive logarithmic Sobolev constant, then it also has a positive entropy constant. The converse is, in general, false.

Moreover, it was also proven in [DS96] that

$$2 \operatorname{gap}(\mathcal{L}_\Lambda^\tau) \geq s(\mathcal{L}_\Lambda^\tau),$$

where the converse is also false in general. Hence, a positive spectral gap is implied by a positive entropy constant, and thus the inequality whose optimal constant is the entropy constant constitutes an inequality in between the logarithmic Sobolev inequality and the Poincaré inequality.

R Remark 3.3.6

We can compare the previous relations between classical constants to the possible relations between their quantum analogues. It is known that a positive quantum logarithmic Sobolev constant (quantum version of the entropy constant here) implies a positive quantum spectral gap [KT16] and also that the former is implied by a positive 2-logarithmic Sobolev constant under the condition of \mathbb{L}_p regularity.

3.4 MIXING CONDITIONS

In this section, we discuss different notions of mixing conditions that need to be assumed on the Gibbs measure to prove that some of the constants introduced in the previous section are positive. More specifically, we will introduce below the notions of *weak* and *strong mixing*, to subsequently compare them and show some of their implications.

First, let us stress that both of them imply that there exists a unique infinite volume Gibbs measure with exponentially decaying variance. Moreover, both notions are essential for the discussion of the exponential ergodicity of a Glauber dynamics for discrete lattice spin systems (the Glauber dynamics is discussed in Section 11.4).

Let us first introduce these notions informally to compare their main differences. For that, consider the Gibbs measure μ_Λ^τ in a lattice Λ with a boundary condition τ and consider $V \subset \Lambda$. On the one side, the *weak mixing condition* implies that a local modification of the boundary condition (at a single site $x \in V^c$) has an influence on the Gibbs measure which decays exponentially fast with the distance from the boundary $\partial^+ V$, whereas the *strong mixing condition* implies, in the same setting, that the influence of the perturbation decays exponentially fast with the distance from the site y .

The difference between both notions is very important, since even for the one phase region (with its unique infinite volume Gibbs measure) with exponentially decaying variance, it might happen that a local perturbation of the boundary condition modifies completely the Gibbs measure close to the boundary, while it leaves the measure essentially unchanged in the bulk. When this effect persists even for V (and thus Λ) arbitrarily large, we refer to this phenomenon as a *boundary phase transition*, and in this situation, it is clear that the Gibbs measure satisfies a weak mixing condition but not a strong one.

R Remark 3.4.1

It is important to highlight that, for certain natural models (such as the Ising model at low temperature and positive external field), the strong mixing condition holds for *regular* volumes, like multiples of a large enough cube, but fails for other sets [MO94a]. From this pathology, a whole revision of the theory of completely analytical Gibbs random fields arose [DS87] and a whole study was carried out to understand which geometries allow for these conditions to hold.

Before introducing formally the two notions of mixing conditions, let us define the projection of a measure. Given $V \subset \Lambda \subset \mathbb{Z}^d$, τ a boundary condition and μ_Λ^τ a Gibbs measure on Ω_Λ , we denote by $\mu_{\Lambda,V}^\tau$ the *projection* of the measure μ_Λ^τ on Ω_V , namely

$$\mu_{\Lambda,V}^\tau(\sigma) := \sum_{\eta: \eta_V = \sigma_V} \mu_\Lambda^\tau(\eta).$$

Then, we can introduce the following two concepts.

Definition 3.4.2 — WEAK MIXING CONDITION, (Mar99).

Given $V \subset \Lambda \subset \mathbb{Z}^d$, τ a boundary condition and μ_Λ^τ a Gibbs measure on Ω_Λ , we say that μ_Λ^τ satisfies the *weak mixing condition* in Λ with constants C and m if for every subset $\Delta \subset V$ the following inequality holds:

$$\sup_{\tau, \tau'} \left\| \mu_{V,\Delta}^\tau - \mu_{V,\Delta}^{\tau'} \right\| \leq C \sum_{x \in \Delta, y \in \partial_r^+ V} e^{-md(x,y)}.$$

Moreover, this condition is denoted by $WM(V, C, m)$.

Definition 3.4.3 — STRONG MIXING CONDITION, (Mar99).

Given $V \subset \Lambda \subset \mathbb{Z}^d$, τ a boundary condition and μ_Λ^τ a Gibbs measure on Ω_Λ , we say that μ_Λ^τ satisfies the *strong mixing condition* in Λ with constants C and m if for every subset $\Delta \subset V$ and every site $y \in V^c$ the following inequality holds:

$$\sup_{\tau} \left\| \mu_{V,\Delta}^\tau - \mu_{V,\Delta}^{\tau^y} \right\| \leq C e^{-md(\Delta,y)},$$

where τ^y coincides with τ at every site except for y .

Moreover, this condition is denoted by $SM(V, C, m)$.

Note that both conditions will be of interest when they hold for a certain dynamics for the same universal constants C and m for an infinite class of finite subsets of \mathbb{Z}^d .

R Remark 3.4.4

As their names suggest, for some cases one can show that the strong condition implies the weak one. Indeed, this is the case at least for all cubes, i.e. strong mixing for all cubes implies weak mixing for all cubes.

The converse is in general expected to be false in dimension greater than two. However, in two dimensions, the following result holds.

Theorem 3.4.5 — WEAK MIXING IMPLIES STRONG MIXING, (MOS94).

In two dimensions, if the condition $WM(V, C, m)$ holds for every $V \subset \mathbb{Z}^d$, then the condition $SM(Q_L, C', m')$ also holds for every square Q_L , for suitable constants C' and m' .

R Remark 3.4.6

Let us emphasize again that the above result becomes false in general when replacing the condition “for all squares” with the condition “for all finite subsets of \mathbb{Z}^2 ”, for instance (see [MO94a]).

Let us discuss the validity of the conditions introduced above for the Ising model. In two dimensions, in the one-phase region, i.e. whenever the external magnetic field h is not null and $\beta < \beta_c$, the condition $WM(V, C, m)$ holds true for any set $V \subset \mathbb{Z}^d$, with constants C and m depending on β and h (see [Hig93], [MO94a] and [SS95]), and by Theorem 3.4.5, the condition $SM(Q_L, C', m')$ also holds for all integers L .

In higher dimensions, for $\beta < \beta_c$ or large enough β and $h \neq 0$, weak mixing also holds [MO94a]. Moreover, for β small enough or βh large enough, strong mixing has also been proven for all cubes.

To conclude this section, let us introduce another form of mixing condition, which will be the one that we will assume for the main result of the next section to hold.

Definition 3.4.7 — MIXING CONDITION, (DPP02).

Given Λ a rectangle of size L and $A, B \subset \Lambda$ of the same size and satisfying $A \cap B = \emptyset$, there exist constants $C_1, C_2 > 0$, depending on β, d and the commuting potential with respect to which the Hamiltonian and thus the Gibbs measure is defined, for which the following condition holds:

$$\sup_{\tau, \sigma \in \Omega} \left| \frac{\mu_{\Lambda}^{\tau}(\eta : \eta_A = \sigma_A) \mu_{\Lambda}^{\tau}(\eta : \eta_B = \sigma_B)}{\mu_{\Lambda}^{\tau}(\eta : \eta_{A \cup B} = \sigma_{A \cup B})} - 1 \right| \leq C_1 e^{-C_2 d(A, B)}. \quad (3.5)$$

This condition can be derived from a condition on the exponential decay of covariances, which can be derived from the *Dobrushin condition* [DS85] (which holds true for β small enough).

3.5 POSITIVE ENTROPY CONSTANT FROM (DPP02)

In this section, we will address the result of positivity of the entropy constant presented in [DPP02], as well as briefly discuss the positivity of the log-Sobolev constant that appears in [Ces01].

In [DPP02], the authors consider a spin system in a finite lattice, whose spins take values in the set of positive integers, and show that, for a certain class of dynamics of this system, under the assumption of a mixing condition in the Gibbs measure associated to this dynamics, there is a positive *entropy constant* (in the quantum setting, we call this notion *modified log-Sobolev constant*, or just *log-Sobolev constant*). For that, they first need to prove a result of quasi-factorization of the entropy of a function in terms of a conditional entropy defined in sub- σ -algebras of the initial σ -algebra.

Let us first recall this notion of conditional entropy.

Definition 3.5.1 — CONDITIONAL ENTROPY.

Given a sub- σ -algebra $\mathcal{G} \subseteq \mathcal{F}$, we define the *conditional entropy* of f in \mathcal{G} by

$$\text{Ent}_{\mu}(f | \mathcal{G}) := \mu(f \log f | \mathcal{G}) - \mu(f | \mathcal{G}) \log \mu(f | \mathcal{G}),$$

where $\mu(f | \mathcal{G})$ is given by

$$\int_G \mu(f | \mathcal{G}) d\mu = \int_G f d\mu \quad \text{for each } G \in \mathcal{G}.$$

With this definition, and the entropy of a function, the following result of quasi-factorization of the entropy can be proven.

Lemma 3.5.2 — QUASI-FACTORIZATION. LEMMAS 5.1 AND 5.2 OF (DPP02).

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and $\mathcal{F}_1, \mathcal{F}_2$ sub- σ -algebras of \mathcal{F} . Suppose that there exists a probability measure $\bar{\mu}$ that makes \mathcal{F}_1 and \mathcal{F}_2 independent, $\mu \ll \bar{\mu}$ and $\mu \mid \mathcal{F}_i = \bar{\mu} \mid \mathcal{F}_i$ for $i = 1, 2$. Then, for every $f \geq 0$ such that $f \log f \in L^1(\mu)$ and $\mu(f) = 1$,

$$\text{Ent}_\mu(f) \leq \frac{1}{1 - 4\|h - 1\|_\infty} \mu \left[\text{Ent}_\mu(f \mid \mathcal{F}_1) + \text{Ent}_\mu(f \mid \mathcal{F}_2) \right],$$

where $h = \frac{d\mu}{d\bar{\mu}}$ is the Radon-Nikodym derivative of μ with respect to $\bar{\mu}$.

In this text, we present a sketch of the proof of this result to compare it with the results of quasi-factorization of the (quantum) relative entropy that will appear in the next part of the thesis.

Proof. First, we can prove

$$\text{Ent}_\mu(f) \leq \mu \left[\text{Ent}_\mu(f \mid \mathcal{F}_1) + \text{Ent}_\mu(f \mid \mathcal{F}_2) \right] + \log \mu[\mu(f \mid \mathcal{F}_1)\mu(f \mid \mathcal{F}_2)],$$

where the last term can be interpreted as a normalization factor that makes possible to write the difference between the RHS and the LHS above as a relative entropy of $f d\mu$ with respect to

$$\frac{\mu(f \mid \mathcal{F}_1)\mu(f \mid \mathcal{F}_2)}{\mu[\mu(f \mid \mathcal{F}_1)\mu(f \mid \mathcal{F}_2)]} d\mu$$

and, thus, conclude that it is positive. The role of this quantity will be played by $\log M$ in all our results of quasi-factorization of Chapter 7, whenever this term appears.

Now, let us upper bound that term by the entropy of f and a multiplicative error term that measures how far \mathcal{F}_1 and \mathcal{F}_2 are from being independent under μ (this measure will be provided by the Radon-Nikodym derivative of μ with respect to $\bar{\mu}$). First, using the fact that \mathcal{F}_1 and \mathcal{F}_2 are independent under $\bar{\mu}$ and the well-known inequality

$$\log(x) \leq x - 1$$

for every $x \geq 0$, we obtain

$$|\log \mu[\mu(f \mid \mathcal{F}_1)\mu(f \mid \mathcal{F}_2)]| \leq |\bar{\mu}[(h - 1)\mu(f \mid \mathcal{F}_1)\mu(f \mid \mathcal{F}_2)]|.$$

Subsequently, we subtract some null terms, use Hölder's inequality and again the fact that \mathcal{F}_1 and \mathcal{F}_2 are independent under $\bar{\mu}$ to get

$$\begin{aligned} & |\bar{\mu}[(h - 1)\mu(f \mid \mathcal{F}_1)\mu(f \mid \mathcal{F}_2)]| \\ & \leq \|h - 1\|_\infty \mu \left(\left| \mu(f \mid \mathcal{F}_1) - \mu \left(\sqrt{\mu(f \mid \mathcal{F}_1)} \right)^2 \right| \right) \mu \left(\left| \mu(f \mid \mathcal{F}_2) - \mu \left(\sqrt{\mu(f \mid \mathcal{F}_2)} \right)^2 \right| \right). \end{aligned}$$

Until this step, the approach followed in the proof of Theorem 5.0.1 can be approximately seen as a quantum version of this one. Moreover, using Cauchy-Schwarz's inequality, the last term can be upper bounded in the following way:

$$\mu \left(\left| \mu(f \mid \mathcal{F}_2) - \mu \left(\sqrt{\mu(f \mid \mathcal{F}_2)} \right)^2 \right| \right) \leq 2 \sqrt{\mu \left(\sqrt{f}, \sqrt{f} \right)}.$$

To conclude, we use the following inequality between the variance of \sqrt{f} and the entropy of f (for which a quantum version can be derived from the quantum Stroock-Varopoulos inequality [BDR18]):

$$\mu(\sqrt{f}, \sqrt{f}) \leq \mu(f \log f).$$

■

These last steps are possibly the most difficult part to extend in the quantum setting. Although there exist both a Cauchy-Schwarz's inequality and a comparison between the variance of the square root of a function and the entropy of the function, these results are not enough to conclude, due to the non-commutativity of the quantum setting.

Now, with this result of quasi-factorization for the entropy and assuming that the Mixing Condition presented in Definition 3.4.7 holds, the positivity of the entropy constant can be proven. However, before that, we consider another result presented in the same paper, which will be necessary to conclude the main result.

Proposition 3.5.3 — (DPP02).

Let $\Lambda \subset \mathbb{Z}^d$ and let \mathcal{L}_Λ^τ be a generator. Then, there is a constant $A > 0$, possibly depending on $|\Lambda|$, but not on $\tau \in \Omega$, such that

$$A^{-1} s(\mathcal{L}_0) \leq s(\mathcal{L}_\Lambda^\tau) \leq A s(\mathcal{L}_0),$$

where \mathcal{L}_0 is the generator associated to the single spin dynamics, which can be seen to have the same entropy constant that the generator of a noninteracting system in which different sites evolve through independent dynamics.

This proposition will be used to reduce the positivity of the entropy constant of the interacting spin system for a certain size to the one of the non-interacting spin system of the same size. The positivity of the latter entropy constant follows from the next proposition, which was also proven in the same article.

Proposition 3.5.4 — (DPP02).

The entropy constant of a non-interacting spin system for a Poisson reference measure of mean λ is lower bounded by λ^{-1} . In particular, it is positive.

Note that the following proposition concerns the Poisson measure, since it is the only measure we will consider in the definition of Equation (3.3). Using these two propositions, we conclude the positivity of the entropy constant for small sublattices to which we reduce the entropy constant of the big lattice in the following result.

Theorem 3.5.5 — POSITIVITY OF THE ENTROPY CONSTANT, (DPP02).

Assume that the Mixing Condition in Definition 3.4.7 holds and consider as the reference measure in Equation (3.3) the Poisson measure. Then, there exists a constant $\alpha > 0$ independent of $|\Lambda|$ and τ such that

$$\alpha \text{Ent}_{\mu_\Lambda^\tau}(f) \leq \mathcal{E}_\Lambda^\tau(f, \log f)$$

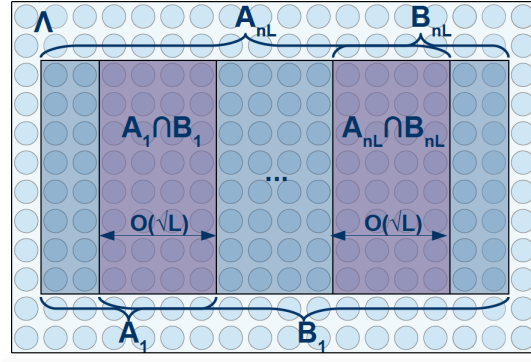
for all $f \geq 0$ so that $f \log f \in \mathbb{L}_1(\mu_\Lambda^\tau)$. In particular, $s(\mathcal{L}_\Lambda^\tau) \geq \alpha > 0$.

The proof of this result consists on a geometric recursive argument to reduce the entropy constant of a big lattice to the one of a small one. This argument is quite similar to the one employed in one part of the proof of Theorem 11.2.2. We sketch below the proof of this result for completeness.

Proof. First, given $L \in \mathbb{N}$, we define

$$s(L) := \inf_{R \in \mathcal{R}_L} \inf_{\tau \in \Omega} s(\mathcal{L}_R^\tau).$$

For every $L \in \mathbb{N}$, one can easily see from Proposition 3.5.3 and Proposition 3.5.4 that every $s(L)$ is positive, although it might be bounded by a positive lower term depending on L . However,

Figure 3.1: Splitting in A_n and B_n .

if we manage to reduce the global entropy constant to $s(L_0)$ for a fixed $L_0 > 0$, we conclude the positivity of the entropy constant.

For that, given a lattice $\Lambda \subset \mathbb{Z}^d$, we split a certain region of the lattice into two families of subregions and we get a lower bound for the entropy constant in terms of the entropy constants in these subregions. Let us construct a suitable family of rectangles in Λ .

Let $R = R(x; l_1, \dots, l_d)$. Without loss of generality, assume that $x = 0$, and $l_1 \leq \dots \leq l_d$. Let us also suppose that $L < l_d \leq 2L$. We define $a_L := \lfloor \sqrt{L} \rfloor$ and $n_L := \lfloor \frac{L}{10a_L} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part. For every integer $1 \leq n \leq n_L$, we cover R with the following pair of rectangles:

$$A_n := \left\{ x \in R : 0 \leq x_d \leq \frac{l_d}{2} + na_L \right\},$$

$$B_n := \left\{ x \in R : \frac{l_d}{2} + (n-1)a_L < x_d \leq l_d \right\}.$$

Hence, for n fixed, it is clear that $A_n \cap B_n \neq \emptyset$ and the shortest side of the overlap has length of order \sqrt{L} (due to the fact that we are considering R a fat rectangle, so $l_1 \geq \frac{1}{10}l_d > \frac{L}{10}$ and if we had $\sqrt{L} > l_1$, we would have $\sqrt{L} > \frac{L}{10}$, or, equivalently, $\frac{L}{100} < 1$, which only holds for L small). See Figure 3.1.

Consider now μ defined as the Gibbs measure and the following two sub- σ -algebras:

$$\mathcal{F}_1 := \sigma\{\eta_i : i \in R \setminus A_n\}, \quad \mathcal{F}_2 := \sigma\{\eta_i : i \in R \setminus B_n\},$$

for every $1 \leq n \leq n_L$. Consider also the following measure, which makes \mathcal{F}_1 and \mathcal{F}_2 independent:

$$\bar{\mu}(\eta) := \mu_{A_n \cap B_n}^{\eta^\tau}(\eta_{A_n \cap B_n}) \mu_{R \setminus A_n}^\tau(\eta_{R \setminus A_n}) \mu_{R \setminus B_n}^\tau(\eta_{R \setminus B_n}).$$

Note that $\mu, \bar{\mu}, \mathcal{F}_1$ and \mathcal{F}_2 satisfy the conditions of Lemma 3.5.2. If we write $h := \frac{d\mu}{d\bar{\mu}}$, it is clear that the Mixing Condition of Definition 3.4.7 implies

$$\|h - 1\|_\infty \leq e^{-C\sqrt{L}}$$

for a certain constant C . Now, for every $f > 0$ such that $f \log f \in \mathbb{L}_1(\mu_R^\tau)$ and by virtue of Lemma 3.5.2 and the previous inequality, we have

$$\text{Ent}_{\mu_R^\tau}(f) \leq \frac{1}{1 - 4e^{-C\sqrt{L}}} \mu_R^\tau \left(\mu_{A_n} \left(f \log \frac{1}{\mu_{A_n}(f)} \right) + \left(f \log \frac{1}{\mu_{B_n}(f)} \right) \right), \quad (3.6)$$

where the notation in the superindex of $\mu_{A_n}^\cdot$ denotes that we are averaging over all possible boundary condition.

From the definition of entropy constant, it is clear that

$$\mu_{A_n}^\sigma \left(f \log \frac{1}{\mu_{A_n}^\sigma(f)} \right) \leq s(\mathcal{L}_{A_n}^\sigma)^{-1} \mathcal{E}_{A_n}^\sigma(f, \log f),$$

and the same for B_n , and thus, replacing it in the expression above, we obtain for every n :

$$\text{Ent}_{\mu_R^\tau}(f) \leq \frac{1}{1 - 4e^{-C\sqrt{L}}} \frac{1}{\inf_{\sigma \in \Omega} \{s(\mathcal{L}_{A_n}^\sigma), s(\mathcal{L}_{B_n}^\sigma)\}} \mu_R^\tau(\mathcal{E}_{A_n}^\tau(f, \log f) + \mathcal{E}_{B_n}^\tau(f, \log f)).$$

Now, after averaging over n , the following inequality holds,

$$s(\mathcal{L}_R^\tau) \geq \left(1 - \frac{\tilde{C}}{\sqrt{L}}\right) \min_n \inf_{\sigma \in \Omega} \{s(\mathcal{L}_{A_n}^\sigma), s(\mathcal{L}_{B_n}^\sigma)\}, \quad (3.7)$$

for a certain positive constant \tilde{C} .

To conclude, it is enough to show that the right hand side can be lower bounded by $s(L)$ multiplied by some constant depending only on L , since this would provide an inequality of the form

$$s(2L) \geq \Psi(L) s(L), \quad (3.8)$$

after taking infimums in the LHS, and thus a recursive procedure on this inequality would allow to reduce the entropy constant on a large lattice to the one of a small sublattice. These two steps are completely analogous to Step 11.2.5 and Step 11.2.6 in Theorem 11.2.2, respectively, but we include a small discussion about them here for completeness.

Let us denote by L_0 the first integer for which inequality (3.8) holds (some of the conditions assumed for the previous reductions need L to be large enough). Let us further consider the expression obtained above and analyze the value of the entropy constant in the rectangles A_n and B_n .

For the rectangle A_n (the analysis is analogous for B_n), we can write it as

$$A_n := x_{A_n} + \left([1, l_1] \times \dots \times [1, l_{d-1}] \times \left[1, \frac{l_d}{2} + na_L\right] \right) \cap \mathbb{Z}^d \quad (3.9)$$

The side corresponding to the coordinate x_d has length less than or equal to $1.2L$, by the definition of A_n . For the other sides, we have to distinguish between two different cases.

1. If $\max \{l_k : k = 1, \dots, d-1\} \leq \frac{3}{2}L$, then the longest side of A_n is less than or equal to $\frac{3}{2}L$, so $A_n \in \mathcal{R}_{\frac{3}{2}L}$ and $s(\mathcal{L}_{A_n}^\sigma) \geq s\left(\frac{3}{2}L\right)$.
2. If the greatest side of A_n , which we call l_i , satisfies $l_i > \frac{3}{2}L$, it is clear that A_n verifies $\max\{l_k\} > 1.5L$ and $\min\{l_k\} \leq 1.2L$. Hence,

$$s(\mathcal{L}_{A_n}^\sigma) \geq \min_{R: \max\{l_k\} > 1.5L, \min\{l_k\} \leq 1.2L} s(\mathcal{L}_R^\sigma).$$

Therefore, for the right-hand side of Equation (3.7), we have

$$\begin{aligned} & \left(1 + \frac{\tilde{C}}{\sqrt{L}}\right)^{-1} \min_{n=1, \dots, n_L} \{s(\mathcal{L}_{A_n}^\sigma), s(\mathcal{L}_{B_n}^\sigma)\} \\ & \geq \left(1 + \frac{\tilde{C}}{\sqrt{L}}\right)^{-1} \min \left\{ s\left(\frac{3}{2}L\right), \min_{R: \max\{l_k\} > 1.5L, \min\{l_k\} \leq 1.2L} s(\mathcal{L}_R^\sigma) \right\}. \end{aligned}$$

Now, we consider a rectangle in \mathcal{R}_{2L} such that its longest side is greater than or equal to $1.5L$ and its shortest side has length less than or equal to $1.2L$. Iterating the procedure carried out to obtain Equation (3.7) at most $d - 1$ times on that rectangle, we end up with a rectangle whose longest side is shorter than or equal to $1.5L$. Hence,

$$\min_{R: \max\{l_k\} > 1.5L, \min\{l_k\} \leq 1.2L} s(\mathcal{L}_R^\sigma) \geq \left(1 + \frac{C}{\sqrt{L}}\right)^{-(d-1)} s\left(\frac{3}{2}L\right).$$

and since the rectangle considered above verified $R \in \mathcal{R}_{2L}$, we obtain

$$s(2L) \geq \left(1 + \frac{\tilde{C}}{\sqrt{L}}\right)^{-d} s\left(\frac{3}{2}L\right).$$

Moreover, if we iterate this expression two more times, we obtain:

$$S(2L) \geq \left(1 + \frac{K}{\sqrt{L}}\right)^{-3d} S(L), \quad (3.10)$$

where K is a constant independent of the size of the system.

Finally, using recursively the relation obtained above, we get a lower bound for the entropy constant in Λ in terms of the entropy constant in small subregions. Indeed, for L_0 as defined above, we have

$$\begin{aligned} \lim_{\Lambda \rightarrow \mathbb{Z}^d} s(\mathcal{L}_\Lambda^\tau) &= \lim_{n \rightarrow \infty} s(2^n L_0) \\ &\geq \left(\prod_{n=1}^{\infty} \left(1 + \frac{K}{\sqrt{2^{n-1} L_0}}\right) \right)^{-3d} s(L_0) \\ &\geq \left(\exp \left[\sum_{n=0}^{\infty} \frac{K}{2^n L_0} \right] \right)^{-3d} s(L_0) \\ &= \exp \left[\frac{-3dK}{L_0} (2 + \sqrt{2}) \right] s(L_0), \end{aligned}$$

where the constants L_0 and K do not depend on the size of Λ . ■

Let us highlight now the main differences of this approach with a possible approach followed in the quantum setting. As mentioned above, the last part of Theorem 3.5.5 completely follows that of Theorem 11.2.2 for quantum spin systems in the context of the Davies dynamics. However, the main difference lies in the first part of the proof of the aforementioned result, and more specifically in the fact that, in the classical case, they average over all possible boundary conditions and reduce the “conditional” terms that appear in the RHS of Equation (3.6) to usual Dirichlet forms due to the existence of the DLR conditions, which do not hold in the quantum case. Therefore, to overcome this issue, we have to introduce in the quantum setting a conditional version of the constant studied (in our case, the log-Sobolev constant) and prove the positivity of this new notion using more elaborate techniques than Propositions 3.5.3 and 3.5.4 above.

R Remark 3.5.6

In [Ces01], a similar result is proven based on an analogous result of quasi-factorization of the entropy. There, the author focuses on the *logarithmic Sobolev inequality* instead of the *entropy inequality*, and shows that, for a Gibbs specification with finite range sumable interaction, the Dobrushin-Shlosman’s complete

analiticity condition implies uniform logarithmic Sobolev inequalities, having the advantage with respect to previous approaches that it relies mostly on properties of the entropy and assumes little on the Dirichlet form.

The specific form of the complete analiticity condition [DS87] that he assumes is the following: There exist $K > 0, m > 0$ such that for all $\Lambda \subset \subset \mathbb{Z}^d, x \in \partial_r^+ \Lambda, V \subset \Lambda$, and for all $\sigma_y = \omega_y$, if $y \neq x$, we have

$$\left\| \frac{\rho_{\Lambda,V}^\omega}{\rho_{\Lambda,V}^\sigma} - 1 \right\|_u \leq K e^{-md(x,\Delta)},$$

where $\rho_{\Lambda,V}^\tau$ is the Radon-Nikodym derivative of $\mu_{\Lambda,V}^\tau$ with respect to a certain measure ν^V .



4. PRELIMINARIES

In this chapter, we review all the basic concepts and properties which are necessary to understand the results presented in the rest of the thesis, with the purpose of creating a self-contained text.

We begin by recalling the notions of *von Neumann entropy* and *relative entropy*, as well as some of their basic properties, in Section 4.1. After collecting some well-known properties and showing the proof of some basic facts, we mention a characterization of the relative entropy that will serve as a basis for the axiomatic characterization of the conditional relative entropy in Chapter 6. In Section 4.2, we recall the definition for Schatten p -norms, as well as for ρ -weighted norms, and show some properties for non-commutative \mathbb{L}_p spaces equipped with any of them.

Afterwards, in Section 4.3, we introduce the notions of conditional expectations and pseudo-conditional expectations, as well as some interesting results concerning them. They will be extremely useful for the development of the rest of the thesis, since the generators of the main dynamics studied here, the Davies and heat-bath dynamics, are associated to a conditional expectation and a pseudo-conditional expectation, respectively. Subsequently, in Section 4.4, we review the notions of operator monotone and operator convex functions, and show some properties that, in particular, also hold for conditional expectations. We will use these properties mainly in Chapter 12.

In the next section, Section 4.5, we introduce the setting of quantum dissipative evolutions used in the whole text and present the notion of log-Sobolev constant. Subsequently, we show how a positive log-Sobolev constant implies a fast convergence of an evolution to its fixed point. Afterwards, in Section 4.6, we recall the concept of Gibbs state and introduce some basic notions related to it. Finally, we review the concept of quantum Markov chains in Section 4.7, in which we recall that Gibbs states are, in particular, quantum Markov chains, and show some results concerning their structure that will be extremely useful, in particular, for Chapter 10.

Before moving to the first section, let us recall that a linear map $\mathcal{T} : \mathcal{B}_\Lambda \rightarrow \mathcal{B}_\Lambda$ is called a superoperator. We write $\mathbb{1}$ for the identity matrix and id for the identity superoperator. For bipartite spaces $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, we consider the natural inclusion $\mathcal{A}_A \hookrightarrow \mathcal{A}_{AB}$ by identifying

This is a picture of Notre Dame, the Cathedral of Paris, a city which I visited first for the *XII Conference on the Theory of Quantum Computation, Communication and Cryptography (TQC 2017)*, in June 2017, and where I also spent almost four months at Institut Henri Poincaré during the *Thematic Trimestre Analysis in Quantum Information*, from September to December 2017.

each operator $f_A \in \mathcal{A}_A$ with $f_A \otimes \mathbb{1}_B$. In this way, we define the modified partial trace in A of $f_{AB} \in \mathcal{A}_{AB}$ by $\text{tr}_A[f_{AB}] \otimes \mathbb{1}_B$, but we denote it by $\text{tr}_A[f_{AB}]$ in a slight abuse of notation. Moreover, we say that an operator $g_{AB} \in \mathcal{A}_{AB}$ has support in A if it can be written as $g_A \otimes \mathbb{1}_B$ for some operator $g_A \in \mathcal{A}_A$. Note that given $f_{AB} \in \mathcal{A}_{AB}$, we write $f_A := \text{tr}_B[f_{AB}]$.

4.1 VON NEUMANN ENTROPY AND RELATIVE ENTROPY

Let us begin this section by recalling the notion of von Neumann entropy and some of its most basic properties.

Definition 4.1.1 — VON NEUMANN ENTROPY.

Let \mathcal{H} be a finite-dimensional Hilbert space, and $\rho \in \mathcal{S}(\mathcal{H})$. The *von Neumann entropy*, or just *quantum entropy*, of ρ is given by:

$$S(\rho) := -\text{tr}[\rho \log \rho]. \quad (4.1)$$

This quantity is widely used in quantum statistical mechanics and is named after John von Neumann. In the following proposition we collect some properties of the von Neumann entropy that will be of use in further sections.

Proposition 4.1.2 — PROPERTIES OF THE VON NEUMANN ENTROPY, (Weh78), (LR73).

Let \mathcal{H}_{AB} be a bipartite finite-dimensional Hilbert space, $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Let $\rho_{AB} \in \mathcal{S}_{AB}$. The following properties hold:

1. **Continuity.** The map $\rho_{AB} \mapsto S(\rho_{AB})$ is continuous.
2. **Nullity.** $S(\rho_{AB})$ is zero if, and only if, ρ_{AB} represents a pure state.
3. **Maximality.** $S(\rho_{AB})$ is maximal, and equal to $\log N$, for $N = \dim(\mathcal{H}_{AB})$, when ρ_{AB} is the maximally mixed state.
4. **Additivity.** $S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B)$.
5. **Subadditivity.** $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$.
6. **Strong subadditivity.** For any three systems A, B and C ,

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}).$$

We introduce now a measure of distinguishability of two states that will be strongly used throughout the whole manuscript, and mention some of its more fundamental properties.

Definition 4.1.3 — RELATIVE ENTROPY, (Ume62).

Let \mathcal{H} be a finite-dimensional Hilbert space, and $\rho, \sigma \in \mathcal{S}(\mathcal{H})$. The *quantum relative entropy* of ρ and σ is given by:

$$D(\rho || \sigma) := \text{tr}[\rho(\log \rho - \log \sigma)]. \quad (4.2)$$

R Remark 4.1.4

In most of this manuscript we only consider density matrices (with trace 1) in the definition of relative entropy. However, it could have been introduced in more generality, for all $f, g \in \mathcal{A}^+$, f verifying $\text{tr}[f] \neq 0$, as follows:

$$D(f || g) = \frac{1}{\text{tr}[f]} \text{tr}[f(\log f - \log g)]. \quad (4.3)$$

Note that we are always considering full-rank operators in these definitions. If the support of the first one is not contained in the support of the second one, the value of the relative entropy is set to be ∞ .

In the next proposition, we can find some well-known properties of the relative entropy.

Proposition 4.1.5 — PROPERTIES OF THE RELATIVE ENTROPY, (Weh78), (LR73).

Let \mathcal{H}_{AB} be a bipartite finite-dimensional Hilbert space, $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Let $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following properties hold:

1. **Continuity.** The map $\rho_{AB} \mapsto D(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Non-negativity.** $D(\rho_{AB}||\sigma_{AB}) \geq 0$ and $D(\rho_{AB}||\sigma_{AB}) = 0 \Leftrightarrow \rho_{AB} = \sigma_{AB}$.
3. **Finiteness.** $D(\rho_{AB}||\sigma_{AB}) < \infty$ if, and only if, $\text{supp}(\rho_{AB}) \subseteq \text{supp}(\sigma_{AB})$, where supp stands for support.
4. **Monotonicity (or data processing inequality).** $D(\rho_{AB}||\sigma_{AB}) \geq D(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB}))$ for every quantum channel \mathcal{T} .
5. **Additivity.** $D(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
6. **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

These properties, especially the property of non-negativity, allow to consider the relative entropy as a measure of separation of two states, even though, technically, it is not a distance (with its usual meaning), since it is not symmetric and lacks a triangle inequality.

Let us prove below the property of superadditivity, whenever $\sigma_{AB} = \sigma_A \otimes \sigma_B$, since it constitutes the starting point of Chapter 5.

Proposition 4.1.6 — SUPERADDITIVITY OF THE RELATIVE ENTROPY.

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. If $\sigma_{AB} = \sigma_A \otimes \sigma_B$, then

$$D(\rho_{AB}||\sigma_{AB}) = I_\rho(A : B) + D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B),$$

where $I_\rho(A : B) = D(\rho_{AB}||\rho_A \otimes \rho_B)$ is the mutual information [Sha48].

As a consequence,

$$D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

Proof. Since $\sigma_{AB} = \sigma_A \otimes \sigma_B$, we have

$$D(\rho_{AB}||\sigma_A \otimes \sigma_B) = \text{tr}[\rho_{AB}(\log \rho_{AB} - \log \sigma_A \otimes \sigma_B)] \quad (4.4)$$

$$= \text{tr}[\rho_{AB}(\log \rho_{AB} - \log \rho_A \otimes \rho_B + \log \rho_A \otimes \rho_B - \log \sigma_A \otimes \sigma_B)] \quad (4.5)$$

$$= D(\rho_{AB}||\rho_A \otimes \rho_B) + D(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) \quad (4.6)$$

$$= I_\rho(A : B) + D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

Now, since $I_\rho(A : B)$ is a relative entropy, it is greater or equal than zero (property 1 of Proposition 4.1.5), so

$$D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

■

Now, using some properties of Propositions 4.1.2 and 4.1.5, one can prove the following well-known result, which will be of use in the following sections. We include a proof for completeness.

Proposition 4.1.7 Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC} \in \mathcal{S}_{ABC}$. Then,

$$I_\rho(A : BC) \geq I_\rho(A : B).$$

Proof. We have

$$I_\rho(A : BC) - I_\rho(A : B) = \text{tr}[\rho_{ABC}(\log \rho_{ABC} - \log \rho_A \otimes \rho_{BC})] \quad (4.7)$$

$$- \text{tr}[\rho_{AB}(\log \rho_{AB} - \log \rho_A \otimes \rho_B)] \quad (4.8)$$

$$= \text{tr}[\rho_{ABC}(\log \rho_{ABC} - \log \rho_A \otimes \rho_{BC} - \log \rho_{AB} + \log \rho_A \otimes \rho_B)] \quad (4.9)$$

$$= \text{tr}[\rho_{ABC}(\log \rho_{ABC} - \log \rho_{BC} - \log \rho_{AB} + \log \rho_B)] \quad (4.10)$$

$$= -S[\rho_{ABC}] + S[\rho_{BC}] + S[\rho_{AB}] - S[\rho_B] \geq 0,$$

where we are using the property of strong subadditivity of Proposition 4.1.2 in the last inequality [LR73]. We are also using the fact that the logarithm of a tensor product is the sum of logarithms (tensored with the identity). ■

The difference between the two terms in the statement of this proposition is called *conditional mutual information*. This result may be seen, hence, as the positivity of this quantity.

We prove now a lemma for observables (non necessarily of trace 1) which yields a relation between the relative entropy of two observables and the relative entropy of some dilations of each of them. In particular, it is a useful tool to express the relative entropy of two observables in terms of the relative entropy of their normalizations (i.e., the quotient of each of them by their trace).

Lemma 4.1.8 Let \mathcal{H} be a finite-dimensional Hilbert space and let $f, g \in \mathcal{A}^+$ such that $\text{tr}[f] \neq 0$. For all positive real numbers a and b , we have:

$$D(af || bg) = D(f || g) + \log \frac{a}{b}. \quad (4.11)$$

Proof. The following chain of identities hold:

$$D(af || bg) = \frac{1}{a \text{tr} f} (a \text{tr}[f(\log af - \log bg)]) \quad (4.12)$$

$$= \frac{1}{\text{tr} f} (\text{tr}[f \log a] + \text{tr}[f \log f] - \text{tr}[f \log b] - \text{tr}[f \log g]) \quad (4.13)$$

$$= \frac{1}{\text{tr} f} (\text{tr}[f(\log f - \log g)]) + \log a - \log b \quad (4.14)$$

$$= D(f || g) + \log \frac{a}{b},$$

where, in the first and third equality, we are using the linearity of the trace, and we are denoting $\log a \mathbb{1}$ by $\log a$ for every $a \geq 0$. ■

Since the relative entropy of two density matrices is non-negative (property 1 of Proposition 4.1.5), we have the following corollary:

Corollary 4.1.9 Let \mathcal{H} be a finite-dimensional Hilbert space and let $f, g \in \mathcal{A}^+$ such that $\text{tr}[f] \neq 0$ and $\text{tr}[g] \neq 0$. Then, the following inequality holds:

$$D(f || g) \geq -\log \frac{\text{tr}[g]}{\text{tr}[f]}. \quad (4.15)$$

Proof. Since $f/\text{tr}[f]$ and $g/\text{tr}[g]$ are density matrices, we have that

$$D(f/\text{tr}[f] || g/\text{tr}[g]) \geq 0,$$

and we can apply Lemma 4.1.8:

$$0 \leq D(f/\text{tr}[f] \parallel g/\text{tr}[g]) = D(f \parallel g) + \log \frac{\text{tr}[g]}{\text{tr}[f]}.$$

■

Let us conclude this section by recalling one of the many axiomatic characterizations of the relative entropy that appears in the literature. We choose the characterization provided by Wilming, Gallego and Eisert [GEW16], building upon work by Matsumoto [Mat10], as it will be of use for the axiomatic characterization of the conditional relative entropy that we will provide in Chapter 6. Before recalling Matsumoto's result, let us introduce the notion of *lower asymptotically semicontinuous* function.

Definition 4.1.10 — LOWER ASYMPTOTICALLY SEMICONTINUITY.

Let \mathcal{H} be a finite-dimensional Hilbert space, $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ and $\{\tilde{\rho}_n\}$ a sequence of states on $\mathcal{H}^{\otimes n}$ for every $n \in \mathbb{N}$. Let f be a function on pairs of quantum states on $\mathcal{H}^{\otimes n}$ for all $n \in \mathbb{N}$. We say that f is *lower asymptotically semicontinuous* with respect to σ if the following condition

$$\lim_{n \rightarrow \infty} \|\rho^{\otimes n} - \tilde{\rho}_n\|_1 = 0$$

implies

$$\liminf_{n \rightarrow \infty} \frac{1}{n} (f(\tilde{\rho}_n, \sigma^{\otimes n}) - f(\rho^{\otimes n}, \sigma^{\otimes n})) \geq 0.$$

Now, we can state the following two results, from which immediately follows the characterization of the relative entropy mentioned above.

Theorem 4.1.11 — (Mat10).

Let f be a function on pairs of quantum states on the same finite-dimensional Hilbert space fulfilling the properties of data processing inequality, additivity and lower asymptotically semicontinuity with respect to every state σ . Then, f is a multiple of the relative entropy.

All the properties mentioned in this and the next results are defined in the same way than their homonyms in Proposition 4.1.5 for the relative entropy.

Lemma 4.1.12 — (GEW16).

Let f be a function on pairs of quantum states on the same finite-dimensional Hilbert space fulfilling the properties of continuity with respect to the first variable, additivity and superadditivity. Then, f is lower asymptotically semicontinuous with respect to every state.

As promised above, from these two results we immediately obtain the following characterization of the relative entropy.

Theorem 4.1.13 — CHARACTERIZATION OF THE RELATIVE ENTROPY, (GEW16).

Let f be a function on pairs of quantum states on the same finite-dimensional Hilbert space fulfilling the properties of continuity with respect to the first variable, data processing inequality, additivity and superadditivity. Then, f is a multiple of the relative entropy.

4.2 NON-COMMUTATIVE \mathbb{L}_p -SPACES

In the following chapters, we will make use of some results concerning Schatten p -norms. Let us introduce this notion and some of their basic properties below.

Definition 4.2.1 — SCHATTEN p -NORMS.

Let \mathcal{H} be a separable Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Given $p \in [1, \infty)$, the Schatten p -norm of T is defined by:

$$\|T\|_p := (\text{tr}[|T|^p])^{1/p},$$

where

$$|T| := \sqrt{T^*T},$$

and T^* is the dual of T with respect to the Hilbert-Schmidt product.

If T is a positive semi-definite operator, we have $\|T\|_p = (\text{tr}[T^p])^{1/p}$. For every $p \in [1, \infty)$, it is a norm, and $\|\cdot\|_\infty := \lim_{p \rightarrow \infty} \|\cdot\|_p$ coincides with the operator norm. For $p = 1$ it is indeed the trace norm. However, for $p < 1$, this is no longer a norm, since it does not satisfy the triangle inequality.

In the following proposition, we collect some basic properties that Schatten p -norms satisfy.

Proposition 4.2.2 — PROPERTIES OF SCHATTEN p -NORMS, (Bha97), (PX03).

Let \mathcal{H} be a separable Hilbert space and $S, T \in \mathcal{B}(\mathcal{H})$. Let $p \in [1, \infty]$, and consider the Schatten p -norm, extending the definition at ∞ by $\|\cdot\|_\infty := \lim_{p \rightarrow \infty} \|\cdot\|_p$ and taking $p = \infty$ as the dual of $q = 1$. The following properties hold:

1. **Monotonicity.** For $1 \leq p \leq p' \leq \infty$, $\|T\|_1 \geq \|T\|_p \geq \|T\|_{p'} \geq \|T\|_\infty$.
2. **Duality.** For $q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $\|S\|_q = \sup\{|\langle S, T \rangle| : \|T\|_p = 1\}$, where $\langle S, T \rangle = \text{tr}[S^*T]$ is the Hilbert-Schmidt inner product.
3. **Unitary invariance.** $\|UTV\|_p = \|T\|_p$ for all unitaries U, V .
4. **Hölder's inequality.** For $q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $\|ST\|_1 \leq \|S\|_p \|T\|_q$.
5. **Sub-multiplicativity.** $\|ST\|_p \leq \|S\|_p \|T\|_p$.

Moreover, some other interesting properties of Schatten p -norms are collected in the following proposition.

Proposition 4.2.3 — MORE PROPERTIES OF SCHATTEN p -NORMS, (PX03).

Let \mathcal{H} be a separable Hilbert space, $S, T \in \mathcal{B}(\mathcal{H})$ and $p \in [1, \infty]$. Consider $0 < r, q \leq \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. The following properties hold:

1. $\|T\|_p = \|T^*\|_p$.
2. **Minkowski's inequality.** $\|T + S\|_p \leq \|T\|_p + \|S\|_p$.
3. **General Hölder's inequality.** $\|TS\|_r \leq \|T\|_p \|S\|_q$.
4. $\|T\|_{2p}^2 = \|TT^*\|_p$.

Now, instead of the Schatten p -norm, one can use a p -weighted inner product to define a non-commutative \mathbb{L}_p space. From this inner product, the following family of weighted norms can be introduced.

Definition 4.2.4 — WEIGHTED NORM, (Kos84).

Let \mathcal{H} be a separable Hilbert space. Given $p \in [1, \infty)$, the ρ -weighted norm which, for a full rank state $\rho \in \mathcal{S}(\mathcal{H})$, is given by

$$\|f\|_{\mathbb{L}_p(\rho)} := \text{tr} \left[\left| \rho^{1/2p} f \rho^{1/2p} \right|^p \right]^{1/p} \text{ for every } f \in \mathcal{A}_{\mathcal{H}}.$$

Analogously, the ρ -weighted inner product (or KMS (Kubo-Martin-Schwinger) inner product) is given by

$$\langle f, g \rangle_{\rho} := \text{tr}[\sqrt{\rho} f \sqrt{\rho} g] \text{ for every } f, g \in \mathcal{A}_{\mathcal{H}}.$$

Some fundamental properties of these spaces are collected in the following proposition.

Proposition 4.2.5 — PROPERTIES OF ρ -WEIGHTED NORMS.

Let $\rho \in \mathcal{S}(\mathcal{H})$. The following properties hold for ρ -weighted norms:

1. **Order.** $\forall p, q \in [1, \infty)$, with $p \leq q$, we have $\|f\|_{\mathbb{L}_p(\rho)} \leq \|f\|_{\mathbb{L}_q(\rho)} \forall f \in \mathcal{A}(\mathcal{H})$.
2. **Duality.** $\forall f \in \mathcal{A}(\mathcal{H})$, we have $\|f\|_{\mathbb{L}_p(\rho)} = \sup \left\{ \langle g, f \rangle_{\rho}, g \in \mathcal{A}(\mathcal{H}), \|g\|_{\mathbb{L}_q(\rho)} \leq 1 \right\}$ for $1/p + 1/q = 1$.
3. **Operator norm.** $\forall f \in \mathcal{A}(\mathcal{H})$, we have $\|f\|_{\mathbb{L}_{\infty}(\rho)} = \|f\|_{\infty}$, the usual operator norm.

The next proposition also collects an important property of this family of spaces which will be used several times on the text. Its proof will be shown in Chapter 5.

Proposition 4.2.6 Let $\rho \in \mathcal{S}(\mathcal{H})$ and consider a completely positive unital linear map $\mathcal{T} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ such that $\mathcal{T}^*(\rho) = \rho$. Then, for any $p \geq 1$ and any $X \in \mathcal{B}(\mathcal{H})$, the following holds:

$$\|\mathcal{T}(X)\|_{\mathbb{L}_p(\rho)} \leq \|X\|_{\mathbb{L}_p(\rho)}.$$

Throughout the whole text, we will use both notions of norms to equip non-commutative \mathbb{L}_p spaces. We will identify in each case to which of these two families of norms we refer. Note that most of the time that a ρ -weighted norm or inner product appears on the text, the role of ρ will be played by the Gibbs state of a local, commuting Hamiltonian.

Let us introduce now another inner product, which differs from the KMS inner product in the lack of symmetry of the position of the weight with respect to the observables where it is being evaluated.

Definition 4.2.7 — GNS INNER PRODUCT.

Let \mathcal{H} be a finite-dimensional Hilbert space and $X, Y \in \mathcal{B}(\mathcal{H})$. Let σ be a state in \mathcal{H} . We define the Gelfand-Naimark-Segal (GNS) inner product of X and Y by

$$\langle X, Y \rangle_{\sigma} := \text{tr}[\sigma X^* Y].$$

To conclude this section, we introduce the notions of variance and covariance of observables, which will also appear frequently in the next chapters, mostly in Chapter 8. From these concepts, the notions of conditional variance and covariance in subsystems will be introduced in the same chapter.

Definition 4.2.8 — COVARIANCE AND VARIANCE.

Let \mathcal{H} be a finite-dimensional Hilbert space, and $\rho \in \mathcal{S}(\mathcal{H})$ a state. Given, $X, Y \in \mathcal{A}(\mathcal{H})$, their *covariance* is defined by

$$\text{Cov}_\rho(X, Y) := |\langle X, Y \rangle - \text{tr}[\rho X] \text{tr}[\rho Y]|.$$

Analogously, we define the variance of X and Y by $\text{Var}_\rho(X) := \text{Cov}_\rho(X, X)$.

4.3 CONDITIONAL EXPECTATIONS

In this section, we turn to conditional expectations. We will first introduce the usual notion of conditional expectations, as it appears e.g. in [OP93], and we will recall some of their most basic (and useful) properties. Afterwards, we will introduce another notion of pseudo-conditional expectations, which will also be of use for us in the next chapters.

Proposition 4.3.1 — CONDITIONAL EXPECTATIONS, (OP93).

Let \mathcal{M} be a matrix algebra with unital matrix subalgebra \mathcal{N} . Then, there exists a unique linear mapping $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ such that

1. \mathcal{E} is a positive map,
2. $\mathcal{E}(B) = B$ for all $B \in \mathcal{N}$,
3. $\mathcal{E}(AB) = \mathcal{E}(A)B$ for all $A \in \mathcal{M}$ and all $B \in \mathcal{N}$,
4. \mathcal{E} is trace preserving.

A map fulfilling (1)-(3) is called a *conditional expectation*.

It can be shown that conditional expectations are completely positive [Ben09] and selfadjoint with respect to the Hilbert-Schmidt inner product. Moreover, given a state σ in \mathcal{M} , the previous notion of conditional expectation can be extended in the following way. A linear mapping $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ is called a *conditional expectation with respect to σ* if conditions (1)-(3) above and the following condition are satisfied:

4. For all $X \in \mathcal{M}$, $\text{tr}[\sigma \mathcal{E}(X)] = \text{tr}[\sigma X]$.

Note that these maps are unital. Furthermore, a conditional expectation satisfies the following useful properties (see [Tak03] for proofs and more details):

Proposition 4.3.2 — PROPERTIES OF CONDITIONAL EXPECTATIONS, (Tak03).

Let \mathcal{M} be a matrix algebra with unital matrix subalgebra \mathcal{N} , σ a density matrix in \mathcal{M} and $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ a conditional expectation with respect to σ . Then, the following properties hold:

1. For all $X \in \mathcal{M}$, $\|\mathcal{E}(X)\| \leq \|X\|$.
2. The following identity holds:

$$\Gamma_\sigma \circ \mathcal{E} = \mathcal{E}^* \circ \Gamma_\sigma,$$

where \mathcal{E}^* denotes the adjoint of \mathcal{E} with respect to the Hilbert-Schmidt inner product and the Γ operator is given by

$$\Gamma_\sigma : \rho \mapsto \sigma^{1/2} \rho \sigma^{1/2}.$$

3. The conditional expectation commutes with the modular automorphism group. Indeed,

$$\Delta_\sigma^{is} \circ \mathcal{E} = \mathcal{E} \circ \Delta_\sigma^{is} \quad \forall s \in \mathbb{R},$$

where the modular operator is given by

$$\Delta_\sigma : \rho \mapsto \sigma \rho \sigma^{-1}.$$

Moreover, given a unital subalgebra $\mathcal{N} \subseteq \mathcal{M}$ and a faithful state σ , the existence of a conditional expectation with respect to σ , $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$, is equivalent to the invariance of \mathcal{N} under the modular automorphism group $(\Delta_\sigma^{is})_{s \in \mathbb{R}}$. Hence, \mathcal{E} is uniquely determined by σ .

Conditional expectations will mostly appear on this text in Chapters 8 and 12. We will sometimes denote $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ by $\mathcal{E}(\cdot | \mathcal{N})$ or $\mathcal{E}_\mathcal{N}$ to emphasize the subalgebra where we are conditioning and to avoid possible mistakes.

Now we turn to introduce a set of maps which we call pseudo-conditional expectations (but appear in the literature as conditional expectations), to highlight the difference with the ones introduced above. We will denote them by \mathbb{E} (see Section 3 of [KB16]).

Definition 4.3.3 — PSEUDO-CONDITIONAL EXPECTATIONS, (KB16).

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite Hilbert space, and σ_{AB} a full-rank state on \mathcal{H}_{AB} . We define a *pseudo-conditional expectation* of σ_{AB} on \mathcal{H}_B by a map $\mathbb{E}_A : \mathcal{H}_{AB} \rightarrow \mathcal{H}_B$ that satisfies the following:

1. **Complete positivity.** \mathbb{E}_A is completely positive and unital.
2. **Consistency.** For every $f_{AB} \in \mathcal{A}_{AB}$,

$$\text{tr}[\sigma_{AB} \mathbb{E}_A(f_{AB})] = \text{tr}[\sigma_{AB} f_{AB}].$$

3. **Reversibility.** For every $f_{AB}, g_{AB} \in \mathcal{A}_{AB}$,

$$\langle \mathbb{E}_A(f_{AB}), g_{AB} \rangle_{\sigma_{AB}} = \langle f_{AB}, \mathbb{E}_A(g_{AB}) \rangle_{\sigma_{AB}}.$$

4. **Monotonicity.** For every $f_{AB} \in \mathcal{A}_{AB}$ and $n \in \mathbb{N}$,

$$\langle \mathbb{E}_A^n(f_{AB}), f_{AB} \rangle_{\sigma_{AB}} \geq \langle \mathbb{E}_A^{n+1}(f_{AB}), f_{AB} \rangle_{\sigma_{AB}}.$$

R Remark 4.3.4

From the properties in the definition of pseudo-conditional expectation, we have:

- Property (2) yields the fact that $\mathbb{E}_A^*(\sigma_{AB}) = \sigma_{AB}$, where the dual is taken with respect to the Hilbert-Schmidt scalar product.
- From property (3) we can deduce that \mathbb{E}_A is self-adjoint in $\mathbb{L}_2(\sigma_{AB})$.

We consider now a specific example of pseudo-conditional expectation. Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\sigma_{AB} \in \mathcal{S}_{AB}$ a full-rank state. We define the *minimal conditional expectation* of $\rho_{AB} \in \mathcal{S}_{AB}$ with respect to σ_{AB} on A by

$$\mathbb{E}_A^\sigma(\rho_{AB}) := \text{tr}_A[\eta_A^\sigma \rho_{AB} \eta_A^{\sigma^\dagger}], \quad (4.16)$$

where $\eta_A^\sigma := (\text{tr}_A[\sigma_{AB}])^{-1/2} \sigma_{AB}^{1/2}$. This map has also been previously called *coarse graining map* and *block spin flip*, among other names [Pet86], [MZ95]. Recalling that $\text{tr}_A[\rho_{AB}] = \rho_B$, we can write

$$\mathbb{E}_A^\sigma(\rho_{AB}) = \sigma_B^{-1/2} \text{tr}_A[\sigma_{AB}^{1/2} \rho_{AB} \sigma_{AB}^{1/2}] \sigma_B^{-1/2}.$$

If we recall now that the partial trace is tensored with the identity in A , we can see that $\mathbb{E}_A^\sigma(\rho)$ is a Hermitian operator and, indeed, \mathbb{E}_A^σ is a pseudo-conditional expectation with respect to σ_{AB} (see [KB16, Proposition 10]). Note that $(\mathbb{E}_A^\sigma)^*$, the adjoint of \mathbb{E}_A^σ with respect to the Hilbert-Schmidt product, which we hereafter denote by \mathbb{E}_A^* to simplify the notation (since we are always considering pseudo-conditional expectations with respect to the same σ_{AB}), is given by

$$\mathbb{E}_A^*(\rho_{AB}) := \sigma_{AB}^{1/2} \sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \sigma_{AB}^{1/2}. \quad (4.17)$$

This map coincides with the Petz recovery map [Pet78] for the partial trace tr_A , composed with the partial trace, and it is a quantum channel. In particular, for every density matrix $\rho_{AB} \in S_{AB}$, $\mathbb{E}_A^*(\rho_{AB})$ is also a density matrix.

This is the only pseudo-conditional expectation we are going to consider in this text hereafter, and we will call it *heat-bath conditional expectation*, since the heat-bath generator is defined after it (see Chapter 10). One should remember that the subscript is used in the same sense as in the partial trace, i.e., denoting the subsystem which is being removed, not the one which is being kept.

4.4 OPERATOR CONVEX FUNCTIONS

Now we will introduce some results concerning operator convex functions that we use in this manuscript, especially in Chapter 12. We refer the reader to [Bha97, Section V] for further information on the topic of operator convex functions. Before introducing operator convex functions, let us first consider operator monotone functions.

Definition 4.4.1 — OPERATOR MONOTONE.

Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval and $f : \mathcal{I} \rightarrow \mathbb{R}$. If for all finite-dimensional Hilbert spaces \mathcal{H}

$$f(A) \leq f(B)$$

for all Hermitian $A, B \in \mathcal{B}(\mathcal{H})$ with spectrum contained in \mathcal{I} and such that $A \leq B$, then f is *operator monotone*. We call f *operator monotone decreasing* if $-f$ is operator monotone.

These functions possess a canonical form that we show in the next result.

Theorem 4.4.2 — (Bha97).

A function f on $(0, \infty)$ is operator monotone if and only if it has a representation of the form

$$f(\lambda) = \alpha + \beta\lambda + \int_0^\infty \left(\frac{t}{1+t^2} - \frac{1}{\lambda+t} \right) d\mu_f(t),$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$ and μ_f is a positive measure on $(0, \infty)$ such that

$$\int_0^\infty \frac{1}{1+t^2} d\mu_f(t) < \infty.$$

Operator monotone functions are intimately connected to operator convex functions. Let us first introduce the latter and show some connections afterwards.

Definition 4.4.3 — OPERATOR CONVEX.

Let $\mathcal{J} \subseteq \mathbb{R}$ be an interval and $f : \mathcal{J} \rightarrow \mathbb{R}$. If

$$f(\lambda A + (1 - \lambda)B) \leq \lambda f(A) + (1 - \lambda)f(B)$$

for all Hermitian $A, B \in \mathcal{B}(\mathcal{H})$ with spectrum contained in \mathcal{J} , all $\lambda \in [0, 1]$, and for all finite-dimensional Hilbert spaces \mathcal{H} , then f is *operator convex*. We call f *operator concave* if $-f$ is operator convex.

One connection of this kind is given by the following theorem:

Theorem 4.4.4 — (Bha97).

Let f be a continuous function mapping $(0, \infty)$ onto itself. Then, f is operator monotone if and only if it is operator concave.

For further connections between operator monotone functions and operator convex functions, we refer to [Bha97, Section V]. Another way to find new operator convex functions is to consider their transpose.

Proposition 4.4.5 — (Bha97).

Let $f : (0, \infty) \rightarrow \mathbb{R}$ and let $\tilde{f}(x) = xf(1/x)$ for all $x \in (0, \infty)$. Then, f is operator convex if and only if \tilde{f} is operator convex. \tilde{f} is called the *transpose* of f .

In the next theorem, we collect several equivalent characterizations of operator convexity. The statements come from [HP03, Theorem 2.1] and [Bha97, Exercise V.2.2].

Theorem 4.4.6 — JENSEN'S OPERATOR INEQUALITY.

For a continuous function f defined on an interval \mathcal{J} , the following conditions are equivalent:

1. f is operator convex on \mathcal{J} .
2. For each natural number n , we have the inequality

$$f\left(\sum_{i=1}^n A_i^* X_i A_i\right) \leq \sum_{i=1}^n A_i^* f(X_i) A_i$$

for every n -tuple (X_1, \dots, X_n) of bounded, self-adjoint operators on an arbitrary Hilbert space \mathcal{H} with spectra contained in \mathcal{J} and every n -tuple (A_1, \dots, A_n) of operators on \mathcal{H} with $\sum_{k=1}^n A_k^* A_k = \mathbb{1}$.

3. $f(V^* X V) \leq V^* f(X) V$ for every Hermitian operator (on a Hilbert space \mathcal{H}) with spectrum in \mathcal{J} and every isometry V from any Hilbert space into \mathcal{H} .

R Remark 4.4.7

Let $\mathcal{J} \subseteq \mathbb{R}$ be an interval, $f : \mathcal{J} \rightarrow \mathbb{R}$ be a continuous operator convex function, \mathcal{M} be a matrix algebra, and $\Phi : \mathcal{M} \rightarrow \mathcal{M}$ a unital completely positive map. Then, Jensen's operator inequality in particular implies that $f(\Phi(X)) \leq \Phi(f(X))$ for any Hermitian $X \in \mathcal{M}$ with spectrum contained in \mathcal{J} . This follows from the fact that any completely positive map possesses a Kraus decomposition.

The following proposition shows that unital positive maps preserve positive definiteness. In particular, this holds for conditional expectations.

Proposition 4.4.8 Let \mathcal{M}, \mathcal{N} be two matrix algebras. Moreover, let $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ be a unital positive map. Then, for $\rho \in \mathcal{M}, \rho > 0$, it holds that $\mathcal{T}(\rho) > 0$.

Proof. Since \mathcal{T} is a positive map, it holds that $\mathcal{T}(\rho) \geq 0$. Assume that $\mathcal{T}(\rho)$ is not positive definite. Then, there is a non-zero $\psi \in \mathcal{H}$ such that $\text{tr}[\psi\psi^* \mathcal{T}(\rho)] = 0$. However,

$$\text{tr}[\psi\psi^* \mathcal{T}(\rho)] = \text{tr}[\mathcal{T}^*(\psi\psi^*)\rho].$$

Since \mathcal{T}^* is also a positive map, $\mathcal{T}^*(\psi\psi^*) \geq 0$. Furthermore, $\mathcal{T}^*(\psi\psi^*) \neq 0$, as \mathcal{T}^* is trace preserving since \mathcal{T} is unital. Hence, $\text{tr}[\mathcal{T}^*(\psi\psi^*)\rho] > 0$, which is a contradiction. ■

Finally, a standard result for completely positive maps which we will use in Section 12.5 is the existence of a Stinespring dilation (see e.g. [Wat18, Theorem 2.22]). It allows us to write a general quantum channel as the action of a conditional expectation $\text{tr}_{\mathcal{V}}[\cdot]/s \otimes I$ and an isometry V .

Theorem 4.4.9 — STINESPRING'S DILATION THEOREM.

Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H}), \mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$ be two matrix algebras with Hilbert spaces \mathcal{H}, \mathcal{K} , and let $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ be a quantum channel. Then, there exist a Hilbert space \mathcal{V} and an isometry $V : \mathcal{H} \hookrightarrow \mathcal{K} \otimes \mathcal{V}$ such that

$$\mathcal{T}(\omega) = \text{tr}_{\mathcal{V}}[V\omega V^*]$$

for all states ω on \mathcal{M} . Here, $\text{tr}_{\mathcal{V}}$ is the partial trace over the second system \mathcal{V} .

4.5 LOG-SOBOLEV INEQUALITIES

In this section, we introduce the notion that lies at the core of this whole thesis, namely log-Sobolev inequalities. For that, we will study open quantum many body systems, which are weakly coupled to an environment. They constitute realistic physical systems and are relevant for quantum information processing. These systems interact with the environment in a considerable way and, thus, the resulting dynamics are dissipative. We shall use for such systems the Markov approximation, which states that the continuous time evolution of a state of such system is given by a quantum Markov semigroup.

Consider a quantum spin lattice system, which will be assumed to live on a d -dimensional finite square lattice, and will be denoted by $\Lambda \subset \mathbb{Z}^d$. To every site x in Λ , we associate a finite-dimensional local Hilbert space \mathcal{H}_x . Then, the Hilbert space associated to the spin lattice Λ is given by $\mathcal{H}_{\Lambda} = \bigotimes_{x \in \Lambda} \mathcal{H}_x$. We denote the set of observables in Λ by \mathcal{A}_{Λ} , and the set of states by \mathcal{S}_{Λ} .

In virtue of the Markov approximation mentioned above, in the Schrödinger picture, given an initial state of the system $\rho_{\Lambda} \in \mathcal{S}_{\Lambda}$, its evolution under the dissipative dynamics is given by a quantum Markov semigroup (QMS), which is nothing but a continuous one-parameter family of linear, CPTP maps (quantum channels, [Wol12]) $\{\mathcal{T}_t\}_{t \geq 0}$ on \mathcal{S}_{Λ} , verifying:

1. $\mathcal{T}_0^* = \mathbb{1}$.
2. $\mathcal{T}_t^* \circ \mathcal{T}_s^* = \mathcal{T}_{t+s}^*$.

The generator of this semigroup is denoted by \mathcal{L}_{Λ}^* , called *Lindbladian* (or *Liouvillian*), since its dual version in the Heisenberg picture satisfies the Lindblad (or GKLS) form [Lin76], [GKS76] for every $X_{\Lambda} \in \mathcal{B}_{\Lambda}$:

$$\mathcal{L}_{\Lambda}(X_{\Lambda}) = i[H, X_{\Lambda}] + \frac{1}{2} \sum_{k=1}^l [2L_k^* X_{\Lambda} L_k - (L_k^* L_k X_{\Lambda} + X_{\Lambda} L_k^* L_k)],$$

where $H \in \mathcal{A}_\Lambda$, the $L_k \in \mathcal{B}_\Lambda$ are the *Lindblad operators* and $[\cdot, \cdot]$ denotes the commutator. Moreover, it is called Liouvillian for satisfying Liouville's equation, i.e.:

$$\frac{d}{dt} \mathcal{T}_t^* = \mathcal{L}_\Lambda^* \circ \mathcal{T}_t^* = \mathcal{T}_t^* \circ \mathcal{L}_\Lambda^*. \quad (4.18)$$

Thus, we can write the elements of the quantum Markov semigroup as

$$\mathcal{T}_t^* = e^{t\mathcal{L}_\Lambda^*}.$$

The notation $*$ appears since we are in the Schrödinger picture, and denotes that this quantum channel may be seen as the dual of another one in the Heisenberg picture. Given $\rho_\Lambda \in \mathcal{S}_\Lambda$, let us denote

$$\rho_t := \mathcal{T}_t^*(\rho_\Lambda)$$

for every $t \geq 0$ (when the omission of the subindex does not cause any confusion). With this notation, Equation (4.18) can be rewritten as the quantum dynamical master equation:

$$\partial_t \rho_t = \mathcal{L}_\Lambda^*(\rho_t).$$

We say that a certain state σ_Λ is an *invariant state* of $\{\mathcal{T}_t^*\}_{t \geq 0}$ if

$$\sigma_t := \mathcal{T}_t^*(\sigma_\Lambda) = \sigma_\Lambda$$

for every $t \geq 0$.

Throughout all this section, we will restrict to the *primitive* case, i.e., $\{\mathcal{T}_t^*\}_{t \geq 0}$ has a unique full-rank invariant state (and thus there is a unique σ_Λ for which $\mathcal{L}_\Lambda^*(\sigma_\Lambda) = 0$). An interesting problem concerning quantum Markov semigroups is the study of the convergence to this unique invariant state, which can be done bounding the mixing time.

The *mixing time* of a quantum Markov semigroup is defined, given an initial state, as the time that the process spends to get close to the invariant state, i.e., the fixed point of the evolution. More specifically, it is given by the following expression

$$\tau(\varepsilon) = \min \left\{ t > 0 : \sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \|\rho_t - \sigma_\Lambda\|_1 \leq \varepsilon \right\}. \quad (4.19)$$

Let us assume that the quantum Markov process studied is *reversible*, i.e., satisfies the *detailed balance condition*

$$\langle f, \mathcal{L}_\Lambda(g) \rangle_{\sigma_\Lambda} = \langle \mathcal{L}_\Lambda(f), g \rangle_{\sigma_\Lambda}$$

for every $f, g \in \mathcal{A}_\Lambda$, where \mathcal{L}_Λ is the generator of the evolution semigroup in the Heisenberg picture.

Different bounds for the mixing time can be obtained by means of the optimal constants for some quantum functional inequalities, such as the *logarithmic Sobolev constant* for the *logarithmic Sobolev inequality* [KT16]. The idea of bounding the mixing time in terms of log-Sobolev constants is based on two facts:

1. Finding a positive functional that bounds the convergence of the semigroup to the fixed point and bounding its derivative in terms of the functional itself. The role of this functional will be played by the relative entropy of ρ_t and σ_Λ :

$$D(\rho_t || \sigma_\Lambda) = \text{tr}[\rho_t (\log \rho_t - \log \sigma_\Lambda)].$$

2. Pinsker's inequality [Pin64]:

$$\|\rho_t - \sigma_\Lambda\|_1 \leq \sqrt{2D(\rho_t || \sigma_\Lambda)}.$$

Let us elaborate this first point. Since ρ_t evolves according to \mathcal{L}_Λ^* , the derivative of $D(\rho_t || \sigma_\Lambda)$ is given by

$$\partial_t D(\rho_t || \sigma_\Lambda) = \text{tr}[\mathcal{L}_\Lambda^*(\rho_t)(\log \rho_t - \log \sigma_\Lambda)],$$

which is a negative quantity (since the relative entropy of ρ_t and σ_Λ decreases with t). For $t = 0$, this quantity is known as the *entropy production*.

Definition 4.5.1 — ENTROPY PRODUCTION.

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice and let \mathcal{H}_Λ be the associated Hilbert space. Let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \rightarrow \mathcal{S}_\Lambda$ be a primitive reversible Lindbladian with fixed point $\sigma_\Lambda \in \mathcal{S}_\Lambda$. Then, for every $\rho_\Lambda \in \mathcal{S}_\Lambda$, the *entropy production* is defined as

$$\text{EP}(\rho_\Lambda) := - \left. \frac{d}{dt} \right|_{t=0} D(\rho_t || \sigma_\Lambda) = - \text{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)].$$

Note that the entropy production of a primitive QMS only vanishes on σ_Λ . The fact that both the negative derivative of the relative entropy between the elements of the semigroup and the fixed point and the relative entropy between the same states have the same kernel and converge to zero with the long time limit, for every possible initial state for the semigroup, allows us to consider the possibility of bounding one in terms of the other, i.e., finding α so that the following holds:

$$2\alpha D(\rho_t || \sigma_\Lambda) \leq \text{EP}(\rho_\Lambda). \quad (4.20)$$

It is clear that, for each ρ_t , there exists an α that makes possible the previous inequality. However, finding a global α that works for every ρ_t is far from trivial. Indeed, in general such quantity does not exist. A global constant for the previous inequality is called a *log-Sobolev constant*.

Definition 4.5.2 — LOG-SOBOLEV CONSTANT.

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice, \mathcal{H}_Λ its associated Hilbert space and $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \rightarrow \mathcal{S}_\Lambda$ a primitive, reversible Lindbladian with fixed point $\sigma_\Lambda \in \mathcal{S}_\Lambda$. We define the *log-Sobolev constant* of \mathcal{L}_Λ^* by

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\text{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

Assume that for a certain Liouvillian \mathcal{L}_Λ^* a positive log-Sobolev constant exists. Then, we can integrate Equation (4.20) to obtain

$$D(\rho_t || \sigma_\Lambda) \leq D(\rho_\Lambda || \sigma_\Lambda) e^{-2\alpha(\mathcal{L}_\Lambda^*)t}, \quad (4.21)$$

and putting this together with Pinsker's inequality, we have:

$$\|\rho_t - \sigma_\Lambda\|_1 \leq \sqrt{2D(\rho_\Lambda || \sigma_\Lambda)} e^{-\alpha(\mathcal{L}_\Lambda^*)t}. \quad (4.22)$$

Finally, for a full-rank state σ_Λ , $D(\rho_\Lambda || \sigma_\Lambda)$ becomes maximal when ρ_Λ corresponds to a rank-one projector onto the minimal eigenvalue of σ_Λ , and thus we obtain:

$$\|\rho_t - \sigma_\Lambda\|_1 \leq \sqrt{2\log(1/\sigma_{\min})} e^{-\alpha(\mathcal{L}_\Lambda^*)t}, \quad (4.23)$$

where σ_{\min} is the minimal eigenvalue of σ_Λ .

Therefore, positive log-Sobolev constants can be used in upper bounds for the mixing time, providing an exponential improvement with respect to a bound in terms of the spectral gap. Indeed, a system for which the mixing time scales polynomially with the system size is said to satisfy *rapid mixing* and this property has profound implications in the system, such as stability against external perturbations [Cub+15] and the fact that its fixed point satisfies an area law for the mutual information [Bra+15a]. Hence, the aforementioned procedure constitutes a way to obtain sufficient conditions for a QMS to satisfy rapid mixing.

Proving whether a Lindbladian has a positive log-Sobolev constant is, thus, a fundamental problem in open quantum many-body systems. We will address later in this text this problem for the heat-bath dynamics and the Davies dynamics, which will be introduced in Section 10.1 and 11.1, respectively.

To conclude this section, let us broadly introduce a couple of notions that will be necessary for the development of the strategy to prove positivity of log-Sobolev constants throughout the rest of the text. First, note that all the specific examples of dynamics, and Lindbladians, considered in this manuscript are *local*, i.e., they can be written as

$$\mathcal{L}_\Lambda := \sum_{x \in \Lambda} \mathcal{L}_x,$$

where \mathcal{L}_x is the Lindbladian on each single site. This property allows to introduce a general definition for the concept of conditional entropy production that is suitable for all our cases of interest, since given a sublattice $A \subseteq \Lambda$, the Lindbladian \mathcal{L}_A is always well-defined as $\sum_{x \in A} \mathcal{L}_x$.

Definition 4.5.3 — CONDITIONAL ENTROPY PRODUCTION.

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice, \mathcal{H}_Λ its associated Hilbert space and $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \rightarrow \mathcal{S}_\Lambda$ a local, primitive, reversible Lindbladian with fixed point $\sigma_\Lambda \in \mathcal{S}_\Lambda$. Given $A \subseteq \Lambda$, we define the *conditional entropy production* for every $\rho_\Lambda \in \mathcal{S}_\Lambda$ by:

$$\text{EP}_A(\rho_\Lambda) := -\text{tr}[\mathcal{L}_A^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)].$$

This notion will be essential to provide a suitable definition for the conditional log-Sobolev constant associated to each dynamics (the first point of the strategy presented in Section 1.2). However, as opposed to the case of the entropy production, a general notion of conditional log-Sobolev constant cannot be provided, since it depends strongly on a conditional relative entropy which, as we will show in Chapter 6, can be introduced in several ways.

Definition 4.5.4 — CONDITIONAL LOG-SOBOLEV CONSTANT.

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice, \mathcal{H}_Λ its associated Hilbert space and $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \rightarrow \mathcal{S}_\Lambda$ a local, primitive, reversible Lindbladian with fixed point $\sigma_\Lambda \in \mathcal{S}_\Lambda$. Given $A \subseteq \Lambda$, we define the *conditional log-Sobolev constant* by:

$$\alpha_\Lambda(\mathcal{L}_A^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\text{tr}[\mathcal{L}_A^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_A(\rho_\Lambda || \sigma_\Lambda)},$$

where $D_A(\rho_\Lambda || \sigma_\Lambda)$ is a conditional relative entropy (see Chapter 6).

4.6 GIBBS STATES

As a continuation of the previous section, let us introduce Gibbs states, which usually play the role of the invariant states of the evolutions mentioned above.

Given a finite lattice $\Lambda \subset \mathbb{Z}^d$, let us define a k -local bounded potential as $\Phi : \Lambda \rightarrow \mathcal{A}_\Lambda$ such that, for any $x \in \Lambda$, $\Phi(x)$ is a Hermitian matrix supported in a ball of radius k centered in x and there exists a constant $C < \infty$ such that $\|\Phi(x)\|_\infty < C$ for every $x \in \Lambda$.

We define the *Hamiltonian* from this potential in the following way: For every subset $A \subset \Lambda$, the Hamiltonian in A , H_A , is given by

$$H_A := \sum_{x \in A} \Phi(x).$$

We further say that this potential is *commuting* if $[\Phi(x), \Phi(y)] = 0$ for every $x, y \in \Lambda$.

Consider now $A \subset \Lambda$ and Φ a bounded k -local potential. Since the potential is local, we can define the (outer) boundary of A as

$$\partial A := \{x \in \Lambda \setminus A \mid d(x, A) < k\}$$

and we denote by $A\partial$ the union of A and its boundary (as we are always considering outer boundaries, we will drop the superindex “+” in ∂ that we introduced in the classical setting). Note that H_A clearly has support in $A\partial$. For the situation of this thesis in which we only focus on 1D systems, for every bounded connected subset $A \subset \Lambda$, the boundary will be composed of two parts, which we will intuitively denote by $(\partial A)_{\text{Left}}$ and $(\partial A)_{\text{Right}}$, respectively.

In the full lattice $\Lambda \subset \mathbb{Z}^d$, the Gibbs state is defined as

$$\sigma_\Lambda := \frac{e^{-\beta H_\Lambda}}{\text{tr}[e^{-\beta H_\Lambda}]}.$$

Note that, by a slight abuse of notations, we will denote by σ_A for $A \subset \Lambda$ the state given by $\text{tr}_{A^c}[\sigma_\Lambda]$, which should not be confused with the restricted Gibbs state corresponding to the terms of the Hamiltonian H_A .

4.7 QUANTUM MARKOV CHAINS

Let us finish the chapter of preliminaries with an introduction to quantum Markov chains. The notions and results that appear on this section will mostly be of use in Chapter 10.

Consider a tripartite space $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. We define a *recovery map* $\mathcal{R}_{B \rightarrow BC}$ from B to BC as a completely positive trace-preserving map that reconstructs the C -part of a state $\sigma_{ABC} \in \mathcal{S}_{ABC}$ from its B -part only. If that reconstruction is possible, i.e., if for a certain $\sigma_{ABC} \in \mathcal{S}_{ABC}$ there exists such $\mathcal{R}_{B \rightarrow BC}$ verifying

$$\sigma_{ABC} = \mathcal{R}_{B \rightarrow BC}(\sigma_{AB}),$$

we say that σ_{ABC} is a *quantum Markov chain* (QMC) between $A \leftrightarrow B \leftrightarrow C$. When this is the case, the recovery map can be taken to be the Petz recovery map:

$$\sigma_{ABC} = \sigma_{BC}^{1/2} \sigma_C^{-1/2} \sigma_{AB} \sigma_C^{-1/2} \sigma_{BC}^{1/2}.$$

This class of states has been deeply studied in the last years. In the next proposition, we collect an equivalent condition for a state to be a QMC in terms of the conditional mutual information.

Theorem 4.7.1 — (Pet86), (Pet03).

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ be a tripartite Hilbert space and $\sigma_{ABC} \in \mathcal{S}_{ABC}$. Then, σ_{ABC} is a quantum Markov chain, if, and only if, $I_\sigma(A : C|B) = 0$, for $I_\sigma(A : C|B) = S(\sigma_{AB}) + S(\sigma_{BC}) - S(\sigma_{ABC}) - S(\sigma_B)$ the quantum conditional mutual information.

Another important equivalent condition for a state to be a quantum Markov chain, concerning its structure as a direct sum of tensor products, appears in the next result.

Theorem 4.7.2 — (Hay+04).

A tripartite state σ_{ABC} of $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ satisfies $I_\sigma(A : C|B) = 0$ if and only if there exists a decomposition of system B as $\mathcal{H}_B = \bigoplus_j \mathcal{H}_{b_j^L} \otimes \mathcal{H}_{b_j^R}$ into a direct sum of tensor products such that

$$\sigma_{ABC} = \bigoplus_j q_j \sigma_{Ab_j^L} \otimes \sigma_{b_j^R C},$$

with the state $\sigma_{Ab_j^L}$ (resp. the state $\sigma_{b_j^R C}$) being on $\mathcal{H}_A \otimes \mathcal{H}_{b_j^L}$ (resp. on $\mathcal{H}_{b_j^R} \otimes \mathcal{H}_C$) and a probability distribution $\{q_j\}$.

Turning now to Gibbs states, as they were introduced in the previous section, we recall a fundamental result about their Markovian structure.

Theorem 4.7.3 — (BP12).

Given a k -local commuting potential on Λ , its associated Gibbs state σ_Λ is a quantum Markov network, that is for all disjoint subsets $A, B, C \subset \Lambda$ such that B shields A from C (in the sense that $\Lambda \setminus B$ is disconnected and A and C lie in two different connected components) with $d(A, C) > k$, $I_\sigma(A : C|B) = 0$.

Therefore, combining the results of Theorem 4.7.3 and Theorem 4.7.2, we obtaining the following essential result for the structure of Gibbs states.

Corollary 4.7.4 Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice and σ_Λ the Gibbs state of a commuting Hamiltonian. Then, for any tripartition $\tilde{A}\tilde{B}\tilde{C}$ of Λ such that \tilde{B} shields \tilde{A} from \tilde{C} , the state σ_Λ can be decomposed as

$$\sigma_\Lambda = \bigoplus_j q_j \sigma_{\tilde{A}\tilde{b}_j^L} \otimes \sigma_{\tilde{b}_j^R \tilde{C}}. \quad (4.24)$$

Using the previous properties for quantum Markov chains, we can easily show the identity of the next proposition.

Proposition 4.7.5 Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ be a tripartite Hilbert space and σ_{ABC} a quantum Markov chain between $A \leftrightarrow B \leftrightarrow C$. Then, the following identity holds:

$$\log \sigma_{ABC} + \log \sigma_B = \log \sigma_{BC} + \log \sigma_{AB}. \quad (4.25)$$

Proof. Since σ_{ABC} is a quantum Markov chain between $A \leftrightarrow B \leftrightarrow C$, by Theorem 4.7.2 we can write it as

$$\sigma_\Lambda = \bigoplus_j q_j \sigma_{Ab_j^L} \otimes \sigma_{b_j^R C}. \quad (4.26)$$

Hence,

$$\begin{aligned}
& -\log \sigma_{ABC} + \log \sigma_{BC} + \log \sigma_{AB} - \log \sigma_B \\
&= \sum_j \left(-\log \sigma_{Ab_j^L} \otimes \sigma_{b_j^R C} + \log \sigma_{b_j^L} \otimes \sigma_{b_j^R C} + \log \sigma_{Ab_j^L} \otimes \sigma_{b_j^R} - \log \sigma_{b_j^L} \otimes \sigma_{b_j^R} \right) \\
&= 0,
\end{aligned}$$

where we have used the fact that the logarithm of a tensor product splits as a sum of logarithms. \blacksquare

As a consequence of this identity, we have the following result.

Corollary 4.7.6 Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ be a tripartite Hilbert space and σ_{ABC} a quantum Markov chain between $A \leftrightarrow B \leftrightarrow C$. Then, for any $\rho_{ABC} \in \mathcal{S}_{ABC}$, the following identity holds:

$$D_A(\rho_{ABC} || \sigma_{ABC}) = D_A(\rho_{AB} || \sigma_{AB}) + I_\rho(A : C|B). \quad (4.27)$$

In particular,

$$D_A(\rho_{ABC} || \sigma_{ABC}) \geq D_A(\rho_{AB} || \sigma_{AB}).$$

Proof. Since σ_{ABC} is a quantum Markov chain between $A \leftrightarrow B \leftrightarrow C$, by Proposition 4.7.5 we have

$$\begin{aligned}
& D_A(\rho_{ABC} || \sigma_{ABC}) - D_A(\rho_{AB} || \sigma_{AB}) \\
&= D(\rho_{ABC} || \sigma_{ABC}) - D(\rho_{BC} || \sigma_{BC}) - D(\rho_{AB} || \sigma_{AB}) + D(\rho_B || \sigma_B) \\
&= \underbrace{-S(\rho_{ABC}) + S(\rho_{BC}) + S(\rho_{AB}) - S(\rho_B)}_{I_\rho(A : C|B)} \\
&\quad + \text{tr}[\rho_{ABC} (-\log \sigma_{ABC} + \log \sigma_{BC} + \log \sigma_{AB} - \log \sigma_B)] \\
&= I_\rho(A : C|B).
\end{aligned}$$

In particular, since $I_\rho(A : C|B) \geq 0$ for every state $\rho_{ABC} \in \mathcal{S}_{ABC}$,

$$D_A(\rho_{ABC} || \sigma_{ABC}) \geq D_A(\rho_{AB} || \sigma_{AB}).$$

Corollary 4.7.7 Consider a tripartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and σ_{ABC} a quantum Markov chain between $A \leftrightarrow B \leftrightarrow C$. Then, for any $\rho_{ABC} \in \mathcal{S}_{ABC}$, the following identity holds:

$$D_A(\rho_{ABC} || \sigma_{ABC}) = -S(\rho_{ABC}) + S(\rho_{BC}) + \text{tr}[\rho_{AB} (-\log \sigma_{AB} + \log \sigma_B)], \quad (4.28)$$

Part II

QUASI-FACTORIZATION OF THE RELATIVE ENTROPY

1st result-piece of the puzzle: Quasi-factorization of the Relative Entropy

This thesis deals with the problem of finding necessary conditions so that a dissipative quantum evolution converges fast enough to its equilibrium. More specifically, we aim to present a strategy that allows us to prove that a quantum Markov semigroup, which models a dissipative quantum system under the Markov approximation, has a positive log-Sobolev constant. As we have discussed in the previous part of the thesis, this strategy will consist of 5 different points, which we can graphically identify as pieces of a puzzle that we ensemble to obtain the positive log-Sobolev constant.

In this part of the thesis, we will focus on the first result-piece of the puzzle¹, the one corresponding to the *quasi-factorization of the relative entropy* (see figure below). In a nutshell, given a certain system, by this notion we mean an upper bound for the relative entropy between two states in that system in terms of some *conditional relative entropies* in certain subsystems and a multiplicative error term with some physical meaning. In the cases we are interested on, it will usually represent a condition often satisfied by a Gibbs state of a Hamiltonian with certain properties.

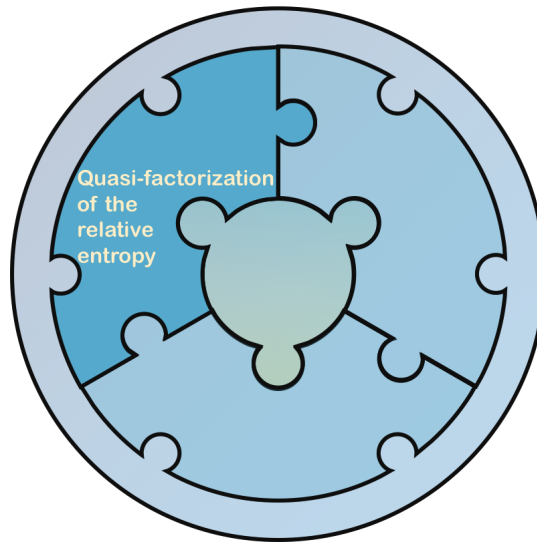


Figure 4.1: Piece of the puzzle corresponding to the quasi-factorization of the relative entropy.

More specifically, we face this part of the manuscript in an increasing order of complexity, both in the notation and the difficulty of the proofs of the results, as well as relevance for the purposes of proving positivity of log-Sobolev constants. We will denote how “close” we are to the ideal result of quasi-factorization by introducing some discontinuity in the lines of the boundary of the piece (see figure below). In this way we will see that we reach the completely continuous lines in the boundary of the piece in the last result of quasi-factorization, as well as the case in which the second state is a tensor product, situations that we call *strong quasi-factorization of the relative entropy*. The main difference with its “weaker” brother mentioned above is that in

¹As mentioned in chapter 1, we split the pieces of the puzzle to prove positivity of a log-Sobolev constant into two classes: *definition-pieces* and *result-pieces*. The former are the inner and outer pieces, and represent two pieces which consist of a proper definition of a certain concept which is used by the latter, the result-pieces, to prove certain results that are necessary for the proof of the global result.

this case we upper bound a conditional relative entropy instead of a relative entropy of two states. This will allow us to use more efficiently in the next part the second result-piece of the puzzle, i.e. the *geometric recursive argument* to reduce global log-Sobolev constants to conditional log-Sobolev constants (and this is how discontinuous boundaries can be interpreted, since with the weaker results it will be more difficult to make the pieces fit).

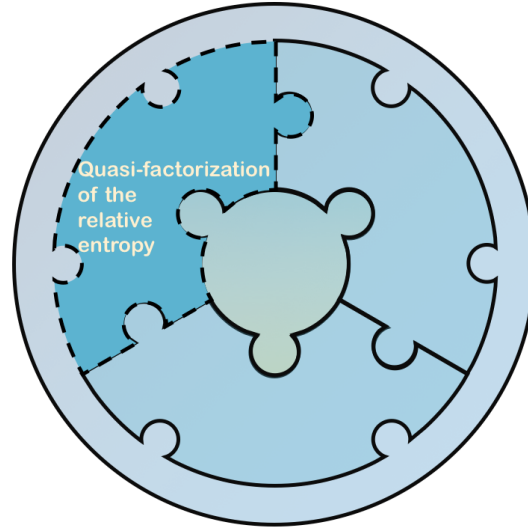


Figure 4.2: Piece of the puzzle corresponding to the (weak) quasi-factorization of the relative entropy.

Therefore, the outline of this part is the following. In Chapter 5 we will present a quantitative extension of the property of superadditivity of the relative entropy to general states, which will turn out to be equivalent to one of the main results of quasi-factorization of the relative entropy of Chapter 7. Before, as we have mentioned above, one of the main characters of these results of quasi-factorization is the conditional relative entropy, a concept that will be introduced and characterized axiomatically in Chapter 6, where two further variants of it will be also introduced for conditional expectations. Subsequently, in Chapter 7, some results of quasi-factorization will be presented in increasing order of difficulty, first using just some properties from the definition of conditional relative entropy and later using the explicit expression for this quantity. Another important result of this family will be presented for *conditional relative entropy by expectations*. Finally, in the last chapter of this part, Chapter 8, we will reach the ideal result we mentioned above, as we will manage to prove a result of strong quasi-factorization for *general conditional relative entropies by expectations*, based on some conditions of *decay of correlations* (corresponding to the outer piece of the puzzle).



5. SUPERADDITIVITY OF THE RELATIVE ENTROPY

In this chapter, we present an extension of the property of superadditivity of the relative entropy to general states. More specifically, recall that the property of *superadditivity* of the relative entropy (Proposition 4.1.5) states that in a bipartite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ one has:

$$D(\rho_{AB} || \sigma_A \otimes \sigma_B) \geq D(\rho_A || \sigma_A) + D(\rho_B || \sigma_B) \quad (5.1)$$

for all $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$, such that $\sigma_{AB} = \sigma_A \otimes \sigma_B$. Then, in a nutshell, the main purpose of this chapter is to provide an inequality in the spirit of (5.1) that holds for every $\sigma_{AB} \in \mathcal{S}_{AB}$ and yields a “measure” of how far this state is from a tensor product.

The interest on this property of the relative entropy lies in its various applications to many different fields, such as statistical physics [OP93, Chapter 13], hypothesis testing [HP91], or even recently in quantum thermodynamics [GEW16]. Indeed, as proven recently in [WGE17] (building on results from [Mat10]), the property of superadditivity, along with the properties of *continuity* with respect to the first variable, *monotonicity* and *additivity* (Proposition 4.1.5), characterizes axiomatically the quantum relative entropy (Theorem 4.1.13).

As mentioned above, the main aim of this chapter is to provide a quantitative extension of (5.1) for an arbitrary density operator σ_{AB} . First, note that for all density matrices ρ_{AB} and σ_{AB} , as a consequence of monotonicity of the quantum relative entropy for the partial trace, the following holds:

$$2D(\rho_{AB} || \sigma_{AB}) \geq D(\rho_A || \sigma_A) + D(\rho_B || \sigma_B). \quad (5.2)$$

Therefore we aim to give a multiplicative term $\alpha(\sigma_{AB}) \in [1, 2]$ at the LHS of (5.1) that measures how far σ_{AB} is from $\sigma_A \otimes \sigma_B$, i.e. an inequality of the form

$$\alpha(\sigma_{AB}) D(\rho_{AB} || \sigma_{AB}) \geq D(\rho_A || \sigma_A) + D(\rho_B || \sigma_B),$$

with $\alpha(\sigma_{AB}) \in [1, 2]$ for every $\sigma_{AB} \in \mathcal{S}_{AB}$.

This result is partially motivated by the quest for results of quasi-factorization of the relative entropy (this topic will be discussed in detail in Chapter 7). Indeed, we will show in Section

This is a picture of Amsterdam that was taken on my way to Delft for the *21th Annual Conference on Quantum Information Processing (QIP 2018)*, in January 2018.

7.3 that a reformulation of Theorem 5.0.1 below constitutes one of the main results of quasi-factorization so far.

Inspired by the work for classical spin systems [DPP02], we will consider $\alpha(\sigma_{AB}) - 1$ as the distance from $\mathbb{1}$ to “ σ_{AB} multiplied by the inverse of $\sigma_A \otimes \sigma_B$ ”. In the classical case, in which σ_{AB} and $\sigma_A \otimes \sigma_B$ commute, there is a unique way to define this, namely $\sigma_{AB}(\sigma_A^{-1} \otimes \sigma_B^{-1})$. However, in the non-commutative case, there are many possible ways to define the multiplication by the inverse. The one we consider in the result below is a symmetric analogue of the commutative case, i.e., $(\sigma_A^{-1/2} \otimes \sigma_B^{-1/2}) \sigma_{AB} (\sigma_A^{-1/2} \otimes \sigma_B^{-1/2})$. Furthermore, another one that will appear in the proof of this result is the derivative of the matrix logarithm on $\sigma_A \otimes \sigma_B$ evaluated on σ_{AB} , $\mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB})$, whose explicit equivalent expressions shown in [Lie73] and [SBT17] will be presented later.

We are now in position to state the main result of this chapter, namely a quantitative extension of the property of superadditivity to general states.

Theorem 5.0.1 — SUPERADDITIVITY OF THE RELATIVE ENTROPY FOR GENERAL STATES, (CLP18b).

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite space. For any bipartite states $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$, the following inequality holds:

$$(1 + 2\|H(\sigma_{AB})\|_\infty)D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B),$$

where

$$H(\sigma_{AB}) = \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB},$$

and $\mathbb{1}_{AB}$ denotes the identity operator in \mathcal{H}_{AB} .

Note that $H(\sigma_{AB}) = 0$ if $\sigma_{AB} = \sigma_A \otimes \sigma_B$.

R Remark 5.0.2

This result constitutes an improvement over (5.2) whenever $\|H(\sigma_{AB})\|_\infty \leq 1/2$ (and, hence, $1 + 2\|H(\sigma_{AB})\|_\infty \leq 2$). Therefore, a tighter statement for Theorem 5.0.1 would be the following: For every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$, the following holds:

$$\min\{2, 1 + 2\|H(\sigma_{AB})\|_\infty\}D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B),$$

with $H(\sigma_{AB})$ defined as above.

Following this way of reasoning, Theorem 5.0.1 is likely to be relevant for situations where it is natural to assume $\sigma_{AB} \sim \sigma_A \otimes \sigma_B$. This is the case of (quantum) many body systems where such property is expected to hold for spatially separated regions A, B in the Gibbs state above the critical temperature.

Indeed, as we saw in Chapter 3, for classical spin systems, a classical version of Theorem 5.0.1 proven by Cesi [Ces01] and Dai Pra, Paganoni and Posta [DPP02] independently and simultaneously, was the key step to notably simplify the proof of the seminal result of Martinelli and Olivieri [MO94b] which connects the decay of correlations in the Gibbs state of a classical spin model with the mixing time of the associated Glauber dynamics, via a bound on the log-Sobolev constant. In Chapter 10, we will use Theorem 5.0.1 and follow these steps to obtain a bound on the mixing time for the quantum heat-bath dynamics via a quantum log-Sobolev constant.

We devote the rest of the chapter to the proof of Theorem 5.0.1. The proof is split into four parts, which constitute the next four sections. The results used in each step are recalled and stated there, for the sake of self-containment.

5.1 STEP 1: ADDITIVE ERROR TERM FOR THE DIFFERENCE OF RELATIVE ENTROPIES

In the first step, we aim to provide a lower bound for the relative entropy of ρ_{AB} and σ_{AB} in terms of $D(\rho_A||\sigma_A)$, $D(\rho_B||\sigma_B)$ and an error term, which we further bound in the following steps.

Step 5.1.1 Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite Hilbert space. For density matrices $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$, it holds that

$$D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B) - \log \operatorname{tr} M, \quad (5.3)$$

where $M = \exp [\log \sigma_{AB} - \log \sigma_A \otimes \sigma_B + \log \rho_A \otimes \rho_B]$ and equality holds, with both sides vanishing, if $\rho_{AB} = \sigma_{AB}$.

Moreover, if $\sigma_{AB} = \sigma_A \otimes \sigma_B$, then $\log \operatorname{tr} M = 0$.

Proof. For the difference of the three relative entropies involved, it clearly holds that:

$$D(\rho_{AB}||\sigma_{AB}) - [D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)] = \quad (5.4)$$

$$= D(\rho_{AB}||\sigma_{AB}) - D(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) \quad (5.5)$$

$$= \operatorname{tr} \left[\rho_{AB} \left(\log \rho_{AB} - \underbrace{(\log \sigma_{AB} - \log \sigma_A \otimes \sigma_B + \log \rho_A \otimes \rho_B)}_{\log M} \right) \right] \quad (5.6)$$

$$= D(\rho_{AB}||M),$$

where M is defined as in the statement of the step and in the first equality we have used the property of additivity of Proposition 4.1.5.

Since M is not a state, the last relative entropy is not necessarily nonnegative. However, we can apply Corollary 4.1.9 to obtain the following inequality:

$$D(\rho_{AB}||M) = \operatorname{tr}[\rho_{AB}(\log \rho_{AB} - \log M)] \geq -\log \operatorname{tr} M.$$

Now, it is easy to check, given the definition of M , that $M = \sigma_{AB}$ if $\rho_{AB} = \sigma_{AB}$, and thus both sides are zero in this case.

Moreover, if $\sigma_{AB} = \sigma_A \otimes \sigma_B$, M is equal to $\rho_A \otimes \rho_B$. In both cases we have $\log \operatorname{tr} M = 0$. ■

5.2 STEP 2: BOUNDING THE ERROR TERM WITH LIEB'S EXTENSION OF GOLDEN-THOMPSON

The aim of the rest of the proof is to bound the additive error term, $\log \operatorname{tr} M$, in terms of the relative entropy between ρ_{AB} and σ_{AB} multiplied by a term that only depends on how far σ_{AB} is from a tensor product. In the second step, we will bound this term by the trace of the product of a term which contains some “distance” between σ_{AB} and $\sigma_A \otimes \sigma_B$ and another one that only depends on ρ_{AB} . For that, we need some previous concepts and results.

First, we recall the Golden-Thompson inequality, proven independently in [Gol65] and [Tho65] (and extended to the infinite-dimensional case in [BG72] and [Rus72]), which states that in a finite-dimensional Hilbert space \mathcal{H} , for every Hermitian operators $f, g \in \mathcal{A}(\mathcal{H})$, the following holds:

$$\operatorname{tr}[e^{f+g}] \leq \operatorname{tr}[e^f e^g], \quad (5.7)$$

where we denote by e^f the exponential of f , defined as

$$e^f := \sum_{k=0}^{\infty} \frac{f^k}{k!}.$$

As Lieb claims in [Lie73], the trivial generalization of the Golden-Thompson inequality to three operators in the form $\text{tr}[e^{f+g+h}] \leq \text{tr}[e^f e^g e^h]$ does not hold in general. However, in the same paper, he provides a correct generalization of this inequality for three operators, which has recently been extended to more operators by Sutter et al. in [SBT17] via the so-called multivariate trace inequalities (in the subsequent paper by Wilde [Wil16], similar inequalities were derived following the statements of [DW16]).

Theorem 5.2.1 — LIEB'S EXTENSION OF GOLDEN-THOMPSON INEQUALITY, (Lie73).

Let $f, g \in \mathcal{A}(\mathcal{H})$ be positive semidefinite operators, and recall the definition of \mathcal{T}_g :

$$\mathcal{T}_g(f) = \int_0^\infty dt (g+t)^{-1} f (g+t)^{-1}. \quad (5.8)$$

Note that \mathcal{T}_g is positive preserving if g is positive. Then, the following inequality holds for every $h \in \mathcal{A}(\mathcal{H})$:

$$\text{tr}[\exp(-f+g+h)] \leq \text{tr}[e^h \mathcal{T}_g(e^f)]. \quad (5.9)$$

This superoperator $\mathcal{T}_g(\cdot)$ constitutes a pseudo-inversion of the operator g with respect to the operator where it is evaluated. In particular, if f and g commute, $\mathcal{T}_g(f)$ is exactly the standard inversion, as we can see in the following corollary.

Corollary 5.2.2 If f and g defined as above commute, then

$$\mathcal{T}_g(f) = f \int_0^\infty dt (g+t)^{-2} = f g^{-1},$$

and therefore

$$\text{tr}[\exp(-f+g+h)] \leq \text{tr}[e^h e^{-f} e^g] = \text{tr}[e^h e^{-f+g}].$$

This shows that Lieb's theorem truly is a generalization of Golden-Thompson inequality.

We use an alternative definition for this superoperator, which allows a straightforward extension of Lieb's result to more than three operators, to obtain a necessary tool for the proof of Step 5.2.5. In [SBT17, Lemma 3.4], Sutter, Berta and Tomamichel proved that Lieb's pseudo-inversion can be rewritten as:

Lemma 5.2.3 — (SBT17).

Let \mathcal{H} be a finite-dimensional Hilbert space and $f, g \in \mathcal{A}(\mathcal{H})$, with g positive semidefinite. Then,

$$\mathcal{T}_g(f) = \int_{-\infty}^\infty dt \beta_0(t) g^{\frac{-1-it}{2}} f g^{\frac{-1+it}{2}},$$

with

$$\beta_0(t) = \frac{\pi}{2} (\cosh(\pi t) + 1)^{-1}.$$

Concerning our problem, we can use this expression for $\mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB})$ to prove the following result, which is a quantum version of a result used in [DPP02]. Note that this could also be proven using the definition for the pseudo-inversion provided by Lieb. The benefit from using the expression of Lemma 5.2.3 will arise later during the proof of Step 5.2.5.

Lemma 5.2.4 — (CLP18b).

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite Hilbert space. For every operator $O_A \in \mathcal{B}_A$ and $O_B \in \mathcal{B}_B$ the following holds:

$$\mathrm{tr}[L(\sigma_{AB}) \sigma_A \otimes O_B] = \mathrm{tr}[L(\sigma_{AB}) O_A \otimes \sigma_B] = 0,$$

where

$$L(\sigma_{AB}) = \mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB}.$$

Proof. We only prove

$$\mathrm{tr}[L(\sigma_{AB}) \sigma_A \otimes O_B] = 0,$$

since the other equality is completely analogous.

Using the expression for $\mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB})$ of Lemma 5.2.3, one can write:

$$\begin{aligned} \mathrm{tr}[L(\sigma_{AB}) \sigma_A \otimes O_B] &= \\ &= \mathrm{tr}[(\mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB}) \sigma_A \otimes O_B] \\ &= \mathrm{tr}[\mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) \sigma_A \otimes O_B] - \mathrm{tr}[\sigma_A \otimes O_B] \\ &= \mathrm{tr}\left[\int_{-\infty}^{\infty} dt \beta_0(t) (\sigma_A \otimes \sigma_B)^{\frac{-1-it}{2}} \sigma_{AB} (\sigma_A \otimes \sigma_B)^{\frac{-1+it}{2}} \sigma_A \otimes O_B\right] - \mathrm{tr}[O_B] \\ &= \int_{-\infty}^{\infty} dt \beta_0(t) \mathrm{tr}\left[\sigma_A^{\frac{-1-it}{2}} \otimes \sigma_B^{\frac{-1-it}{2}} \sigma_{AB} \sigma_A^{\frac{-1+it}{2}} \otimes \sigma_B^{\frac{-1+it}{2}} \sigma_A \otimes O_B\right] - \mathrm{tr}[O_B], \end{aligned}$$

since $\mathrm{tr}[\sigma_A] = 1$, the integral commutes with the trace, $\beta_0(t)$ is a scalar for every $t \in \mathbb{R}$ and the exponent in the power of a tensor product can be split into both terms.

Now, since the trace is cyclic and using the fact that any operator in \mathcal{H}_B commutes with every operator in \mathcal{H}_A , we have:

$$\begin{aligned} \mathrm{tr}[L(\sigma_{AB}) \sigma_A \otimes O_B] &= \\ &= \int_{-\infty}^{\infty} dt \beta_0(t) \mathrm{tr}\left[\sigma_{AB} \sigma_A^{\frac{-1+it}{2}} \otimes \sigma_B^{\frac{-1+it}{2}} \sigma_A \otimes O_B \sigma_A^{\frac{-1-it}{2}} \otimes \sigma_B^{\frac{-1-it}{2}}\right] - \mathrm{tr}[O_B] \\ &= \int_{-\infty}^{\infty} dt \beta_0(t) \mathrm{tr}\left[\sigma_{AB} \left(\sigma_A^{\frac{-1+it}{2}} \sigma_A \sigma_A^{\frac{-1-it}{2}}\right) \otimes \left(\sigma_B^{\frac{-1+it}{2}} O_B \sigma_B^{\frac{-1-it}{2}}\right)\right] - \mathrm{tr}[O_B] \\ &= \int_{-\infty}^{\infty} dt \beta_0(t) \mathrm{tr}\left[\sigma_{AB} \mathbb{1}_A \otimes \left(\sigma_B^{\frac{-1+it}{2}} O_B \sigma_B^{\frac{-1-it}{2}}\right)\right] - \mathrm{tr}[O_B] \\ &= \int_{-\infty}^{\infty} dt \beta_0(t) \mathrm{tr}\left[\sigma_B \sigma_B^{\frac{-1+it}{2}} O_B \sigma_B^{\frac{-1-it}{2}}\right] - \mathrm{tr}[O_B] \\ &= \int_{-\infty}^{\infty} dt \beta_0(t) \mathrm{tr}\left[\sigma_B^{\frac{-1-it}{2}} \sigma_B \sigma_B^{\frac{-1+it}{2}} O_B\right] - \mathrm{tr}[O_B] \\ &= \mathrm{tr}[O_B] \int_{-\infty}^{\infty} dt \beta_0(t) - \mathrm{tr}[O_B] \\ &= 0, \end{aligned}$$

where we have used

$$\int_{-\infty}^{\infty} dt \beta_0(t) = 1,$$

and the fact that, for every observable $f_A \in \mathcal{B}_A$ and state $\rho_{AB} \in \mathcal{S}_{AB}$, the following holds:

$$\mathrm{tr}[f_A \otimes \mathbb{1}_B \rho_{AB}] = \mathrm{tr}[f_A \rho_A].$$

■

We are now in position to state and prove the second step of the proof of Theorem 5.0.1.

Step 5.2.5 Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite Hilbert space. For density matrices $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$, it holds that

$$\log \operatorname{tr} M \leq \operatorname{tr}[L(\sigma_{AB})(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)], \quad (5.10)$$

where

$$L(\sigma_{AB}) = \mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB}.$$

Proof. We apply Lieb's theorem to the error term of inequality (5.3):

$$\operatorname{tr} M = \operatorname{tr} \left[\exp \left(\underbrace{\log \sigma_{AB}}_g - \underbrace{\log \sigma_A \otimes \sigma_B}_f + \underbrace{\log \rho_A \otimes \rho_B}_h \right) \right] \quad (5.11)$$

$$\leq \operatorname{tr}[\rho_A \otimes \rho_B \mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB})] \quad (5.12)$$

$$= \operatorname{tr} \left[\rho_A \otimes \rho_B \underbrace{(\mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB})}_{L(\sigma_{AB})} \right] + \underbrace{\operatorname{tr}[\rho_A \otimes \rho_B]}_1,$$

where we are adding and subtracting $\rho_A \otimes \rho_B$ inside the trace in the last equality.

Now, using the well-known fact $\log(x) \leq x - 1$, we have

$$\log \operatorname{tr} M \leq \operatorname{tr} M - 1 \leq \operatorname{tr}[L(\sigma_{AB})\rho_A \otimes \rho_B].$$

Finally, by virtue of Lemma 5.2.4, it is clear that

$$\operatorname{tr}[L(\sigma_{AB})\rho_A \otimes \rho_B] = \operatorname{tr}[L(\sigma_{AB})(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)].$$

Therefore,

$$\log \operatorname{tr} M \leq \operatorname{tr}[L(\sigma_{AB})(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)].$$

Note that if $\sigma_{AB} = \sigma_A \otimes \sigma_B$, we still get back a null error term, since $\mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) = (\sigma_A \otimes \sigma_B)^{-1} \sigma_A \otimes \sigma_B = \mathbb{1}_{AB}$, and thus $L(\sigma_{AB}) = 0$. This yields the fact that we are not loosing too much in these bounds. ■

5.3 STEP 3: HÖLDER'S AND PINSKER'S INEQUALITIES TO GET BACK A RELATIVE ENTROPY

In the third step of the proof of Theorem 5.0.1, we bound $\operatorname{tr}[L(\sigma_{AB})(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)]$ by the relative entropy of ρ_{AB} and σ_{AB} multiplied by an expression depending only on $L(\sigma_{AB})$, which has the spirit of the multiplicative error term we want to have at the end, since $L(\sigma_{AB})$ represents how far σ_{AB} is from a tensor product between the regions A and B .

The first well-known result we recall for this step is Pinsker's inequality.

Theorem 5.3.1 — PINSKER'S INEQUALITY, (Csi67; Pin64).

For ρ_{AB} and σ_{AB} density matrices on a bipartite Hilbert space \mathcal{H}_{AB} , it holds that

$$\|\rho_{AB} - \sigma_{AB}\|_1^2 \leq 2D(\rho_{AB}||\sigma_{AB}). \quad (5.13)$$

This result will be of use at the end of the proof to finally obtain the relative entropy in the right-hand side of the desired inequality. However, one important thing to highlight is the different orders between the \mathbb{L}_1 -norm of the difference between ρ_{AB} and σ_{AB} and the relative entropy of ρ_{AB} and σ_{AB} in Pinsker's inequality. The exponent we need for the relative entropy is one, and from an \mathbb{L}_1 -norm and Pinsker's inequality we would only get $1/2$, thus we will need to

increase the degree of the term with the trace we already have and from which we will construct an \mathbb{L}_1 -norm.

We will see later that the fact that in $\text{tr}[L(\sigma_{AB})(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)]$ the two factors $(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)$ appear, the multiplicativity of the trace with respect to tensor products and the monotonicity of the relative entropy play a decisive role in the proof.

Another important fact that we note in the left-hand side of Pinsker's inequality is that there is a difference between two states (in fact, the ones appearing in the relative entropy). This justifies the use of Lemma 5.2.4 at the end of Step 5.2.5, to obtain something similar to the difference between ρ_{AB} and σ_{AB} .

We are now ready to prove the third step in the proof of Theorem 5.0.1.

Step 5.3.2 Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite Hilbert space. For density matrices $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$, the following inequality holds:

$$\text{tr}[L(\sigma_{AB})(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)] \leq 2\|L(\sigma_{AB})\|_\infty D(\rho_{AB}||\sigma_{AB}). \quad (5.14)$$

Proof. We use the multiplicativity with respect to tensor products of the trace norm and Hölder's inequality between the trace norm and the operator norm to get:

$$\begin{aligned} \text{tr}[L(\sigma_{AB})(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)] &\leq \|L(\sigma_{AB})\|_\infty \|(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)\|_1 \\ &= \|L(\sigma_{AB})\|_\infty \|\rho_A - \sigma_A\|_1 \|\rho_B - \sigma_B\|_1. \end{aligned} \quad (5.15)$$

Finally, Pinsker's inequality (Theorem 5.3.1) implies that

$$\|\rho_A - \sigma_A\|_1 \leq \sqrt{2D(\rho_A||\sigma_A)}, \quad \|\rho_B - \sigma_B\|_1 \leq \sqrt{2D(\rho_B||\sigma_B)}.$$

Therefore,

$$\|\rho_A - \sigma_A\|_1 \|\rho_B - \sigma_B\|_1 \leq 2\sqrt{D(\rho_A||\sigma_A)D(\rho_B||\sigma_B)} \leq 2D(\rho_{AB}||\sigma_{AB}),$$

where in the last inequality we have used monotonicity of the relative entropy with respect to the partial trace (Proposition 4.1.5). ■

5.4 STEP 4: NON-COMMUTATIVE \mathbb{L}_p -NORMS TO GET A NICER ERROR TERM

Note that if we put together Steps 5.1.1, 5.2.5 and 5.3.2, we obtain the following expression for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$:

$$(1 + 2\|L(\sigma_{AB})\|_\infty)D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B), \quad (5.16)$$

with

$$L(\sigma_{AB}) = \mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB}.$$

This inequality already constitutes a quantitative extension of (5.1) for arbitrary density matrices σ_{AB} in the following sense: If σ_{AB} is a tensor product between A and B , we recover the usual superadditivity, and in general $\|L(\sigma_{AB})\|_\infty$ gives a “distance” between σ_{AB} and $\sigma_A \otimes \sigma_B$.

In the fourth and final step of the proof, we bound $\|L(\sigma_{AB})\|_\infty$ by

$$\left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty,$$

a quantity from which the closeness to 0 whenever σ_{AB} is close to be a tensor product is directly deduced. It also has some physical interpretation in quantum many body systems that will be discussed after proving Step 5.4.2, in the next section.

Recalling non-commutative weighted \mathbb{L}_p -spaces, introduced in Section 4.2, we can prove the following tool, which will of use in the proof of Step 5.4.2.

Lemma 5.4.1 — (CLP18b).

Consider $\rho \in \mathcal{S}_{AB}$ and let T be a quantum channel verifying $T^*(\rho) = \rho$, for T^* the dual of T with respect to the Hilbert-Schmidt scalar product. Then, T is contractive between $\mathbb{L}_1(\rho)$ and $\mathbb{L}_1(\rho)$, i.e., the following inequality holds for every $X \in \mathcal{B}_{AB}$:

$$\|T(X)\|_{\mathbb{L}_1(\rho)} \leq \|X\|_{\mathbb{L}_1(\rho)}. \quad (5.17)$$

Proof. Using the property of duality for the ρ -weighted norms of \mathbb{L}_p -spaces (property 2 of Proposition 4.2.5), we can write:

$$\begin{aligned} \|T(X)\|_{\mathbb{L}_1(\rho)} &= \sup_{\|Y\|_{\mathbb{L}_\infty(\rho)} \leq 1} \operatorname{tr} \left[T(X) \rho^{1/2} Y \rho^{1/2} \right] \\ &= \sup_{\|Y\|_{\infty} \leq 1} \operatorname{tr} \left[T(X) \rho^{1/2} Y \rho^{1/2} \right] \\ &= \sup_{-\mathbb{1} \leq Y \leq \mathbb{1}} \operatorname{tr} \left[T(X) \rho^{1/2} Y \rho^{1/2} \right], \end{aligned}$$

where in the first equality we have used the fact that, for every $\rho \in \mathcal{S}_{AB}$, $\|\cdot\|_{\mathbb{L}_\infty(\rho)}$ coincides with the operator norm.

Recalling now that T^* is the dual of T with respect to the Hilbert-Schmidt scalar product, we have:

$$\begin{aligned} \operatorname{tr} \left[T(X) \rho^{1/2} Y \rho^{1/2} \right] &= \operatorname{tr} \left[X T^*(\rho^{1/2} Y \rho^{1/2}) \right] \\ &= \operatorname{tr} \left[X \rho^{1/2} \rho^{-1/2} T^*(\rho^{1/2} Y \rho^{1/2}) \rho^{-1/2} \rho^{1/2} \right]. \end{aligned}$$

Since we are considering the supremum over the observables verifying $-\mathbb{1} \leq Y \leq \mathbb{1}$, if we apply to these inequalities $T^*(\rho^{1/2} \cdot \rho^{1/2})$, we have $-\rho \leq T^*(\rho^{1/2} Y \rho^{1/2}) \leq \rho$ (because of the assumption $T^*(\rho) = \rho$).

Hence, if we write $Z = \rho^{-1/2} T^*(\rho^{1/2} Y \rho^{1/2}) \rho^{-1/2}$, it is clear that whenever $-\mathbb{1} \leq Y \leq \mathbb{1}$ holds, $-\mathbb{1} \leq Z \leq \mathbb{1}$ also does. Therefore,

$$\begin{aligned} \|T(X)\|_{\mathbb{L}_1(\rho)} &= \sup_{-\mathbb{1} \leq Y \leq \mathbb{1}} \operatorname{tr} \left[T(X) \rho^{1/2} Y \rho^{1/2} \right] \\ &= \sup_{-\mathbb{1} \leq Y \leq \mathbb{1}} \operatorname{tr} \left[X \rho^{1/2} \rho^{-1/2} T^*(\rho^{1/2} Y \rho^{1/2}) \rho^{-1/2} \rho^{1/2} \right] \\ &\leq \sup_{-\mathbb{1} \leq Z \leq \mathbb{1}} \operatorname{tr} \left[X \rho^{1/2} Z \rho^{1/2} \right] \\ &= \|X\|_{\mathbb{L}_1(\rho)}, \end{aligned}$$

where the last equality comes again from the property of duality of weighted \mathbb{L}_p -norms. ■

In the proof of the previous lemma we have strongly used the property of duality of $\mathbb{L}_p(\rho)$. Indeed, the fact that the $\mathbb{L}_1(\rho)$ -norm is the dual of the operator norm has been essential to obtain the desired result. Using similar tools, we can now prove the last step in the proof of Theorem 5.0.1.

Step 5.4.2 With the notation of the previous steps, we have

$$\|L(\sigma_{AB})\|_\infty \leq \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty. \quad (5.18)$$

Proof. The strategy we follow in this proof is the inverse to the one used in the previous lemma, i.e., we study now the $\mathbb{L}_\infty(\sigma_A \otimes \sigma_B)$ -norm as the dual of the $\mathbb{L}_1(\sigma_A \otimes \sigma_B)$ -norm. Since $\|\cdot\|_{\mathbb{L}_\infty(\rho_{AB})}$ coincides with the usual ∞ -norm (operator norm) for every $\rho_{AB} \in \mathcal{S}_{AB}$, we can write

$$\|L(\sigma_{AB})\|_\infty = \|\mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB}\|_{\mathbb{L}_\infty(\sigma_A \otimes \sigma_B)}.$$

Using the aforementioned property of duality for the $\sigma_A \otimes \sigma_B$ -weighted norms of \mathbb{L}_p -spaces, we have:

$$\begin{aligned} \|\mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB}\|_{\mathbb{L}_\infty(\sigma_A \otimes \sigma_B)} &= \\ &= \sup_{\|O_{AB}\|_{\mathbb{L}_1(\sigma_A \otimes \sigma_B)} \leq 1} \langle O_{AB}, \mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB} \rangle_{\sigma_A \otimes \sigma_B} \\ &= \sup_{\|O_{AB}\|_{\mathbb{L}_1(\sigma_A \otimes \sigma_B)} \leq 1} \text{tr} \left[(\sigma_A \otimes \sigma_B)^{1/2} O_{AB} (\sigma_A \otimes \sigma_B)^{1/2} (\mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB}) \right] \\ &= \sup_{\|O_{AB}\|_{\mathbb{L}_1(\sigma_A \otimes \sigma_B)} \leq 1} \left(\underbrace{\text{tr} \left[\sigma_A^{1/2} \otimes \sigma_B^{1/2} O_{AB} \sigma_A^{1/2} \otimes \sigma_B^{1/2} \mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) \right]}_R \right. \\ &\quad \left. - \underbrace{\text{tr} \left[\sigma_A^{1/2} \otimes \sigma_B^{1/2} O_{AB} \sigma_A^{1/2} \otimes \sigma_B^{1/2} \right]}_S \right). \end{aligned}$$

Let us analyze the terms R and S separately. Concerning R , we can write it as the trace with respect to σ_{AB} of a twirled observable as follows:

$$\begin{aligned} R &= \text{tr} \left[\sigma_A^{1/2} \otimes \sigma_B^{1/2} O_{AB} \sigma_A^{1/2} \otimes \sigma_B^{1/2} \mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) \right] \\ &= \text{tr} \left[(\sigma_A \otimes \sigma_B)^{1/2} O_{AB} (\sigma_A \otimes \sigma_B)^{1/2} \int_{-\infty}^{\infty} dt \beta_0(t) (\sigma_A \otimes \sigma_B)^{-\frac{1-i}{2}} \sigma_{AB} (\sigma_A \otimes \sigma_B)^{-\frac{1+i}{2}} \right] \\ &= \text{tr} \left[O_{AB} \int_{-\infty}^{\infty} dt \beta_0(t) (\sigma_A \otimes \sigma_B)^{-\frac{i}{2}} \sigma_{AB} (\sigma_A \otimes \sigma_B)^{\frac{i}{2}} \right] \\ &= \int_{-\infty}^{\infty} dt \beta_0(t) \text{tr} \left[O_{AB} (\sigma_A \otimes \sigma_B)^{-\frac{i}{2}} \sigma_{AB} (\sigma_A \otimes \sigma_B)^{\frac{i}{2}} \right] \\ &= \int_{-\infty}^{\infty} dt \beta_0(t) \text{tr} \left[(\sigma_A \otimes \sigma_B)^{\frac{i}{2}} O_{AB} (\sigma_A \otimes \sigma_B)^{-\frac{i}{2}} \sigma_{AB} \right] \\ &= \text{tr} \left[\underbrace{\sigma_{AB} \int_{-\infty}^{\infty} dt \beta_0(t) (\sigma_A \otimes \sigma_B)^{\frac{i}{2}} O_{AB} (\sigma_A \otimes \sigma_B)^{-\frac{i}{2}}}_{\tilde{O}_{AB}} \right], \end{aligned}$$

where in the third and last equality we have used the fact that the integral and the trace commute, and the fourth equality is due to the cyclicity of the trace. We have also defined:

$$\tilde{O}_{AB} := \int_{-\infty}^{\infty} dt \beta_0(t) (\sigma_A \otimes \sigma_B)^{\frac{i}{2}} O_{AB} (\sigma_A \otimes \sigma_B)^{-\frac{i}{2}}.$$

Now, writing S in terms of \tilde{O}_{AB} would allow us to simplify the expression that appears in the supremum above. We can do that in the following way:

$$\begin{aligned}
S &= \text{tr} \left[\sigma_A^{1/2} \otimes \sigma_B^{1/2} O_{AB} \sigma_A^{1/2} \otimes \sigma_B^{1/2} \right] \\
&= \text{tr} \left[\sigma_A^{1/2} \otimes \sigma_B^{1/2} O_{AB} \sigma_A^{1/2} \otimes \sigma_B^{1/2} \int_{-\infty}^{\infty} dt \beta_0(t) \right] \\
&= \int_{-\infty}^{\infty} dt \beta_0(t) \text{tr} \left[\sigma_A^{1/2} \otimes \sigma_B^{1/2} O_{AB} \sigma_A^{1/2} \otimes \sigma_B^{1/2} \right] \\
&= \int_{-\infty}^{\infty} dt \beta_0(t) \text{tr} \left[(\sigma_A \otimes \sigma_B) (\sigma_A \otimes \sigma_B)^{\frac{i}{2}} O_{AB} (\sigma_A \otimes \sigma_B)^{-\frac{i}{2}} \right] \\
&= \text{tr} \left[(\sigma_A \otimes \sigma_B) \int_{-\infty}^{\infty} dt \beta_0(t) (\sigma_A \otimes \sigma_B)^{\frac{i}{2}} O_{AB} (\sigma_A \otimes \sigma_B)^{-\frac{i}{2}} \right] \\
&= \text{tr} \left[(\sigma_A \otimes \sigma_B) \tilde{O}_{AB} \right],
\end{aligned}$$

where we have used again the properties of cyclicity of the trace and commutativity of the integral and the trace.

Replacing now the values for R and S that we have just computed in the supremum of the first part of the proof, we have:

$$\begin{aligned}
\| \mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB} \|_{\mathbb{L}_\infty(\sigma_A \otimes \sigma_B)} &= \sup_{\|O_{AB}\|_{\mathbb{L}_1(\sigma_A \otimes \sigma_B)} \leq 1} \left(\text{tr} \left[\sigma_{AB} \tilde{O}_{AB} \right] - \text{tr} \left[\sigma_A \otimes \sigma_B \tilde{O}_{AB} \right] \right) \\
&= \sup_{\|O_{AB}\|_{\mathbb{L}_1(\sigma_A \otimes \sigma_B)} \leq 1} \text{tr} \left[\tilde{O}_{AB} (\sigma_{AB} - \sigma_A \otimes \sigma_B) \right].
\end{aligned}$$

This expression looks much simpler than the one we had before. However, we need to prove that $\| \tilde{O}_{AB} \|_{\mathbb{L}_1(\sigma_A \otimes \sigma_B)} \leq 1$ so that \tilde{O}_{AB} is one of the terms of the set where the supremum is evaluated. Indeed, if we consider the map $T : \mathcal{A}_{AB} \rightarrow \mathcal{A}_{AB}$ given by

$$O_{AB} \mapsto \int_{-\infty}^{\infty} dt \beta_0(t) (\sigma_A \otimes \sigma_B)^{\frac{i}{2}} O_{AB} (\sigma_A \otimes \sigma_B)^{-\frac{i}{2}},$$

it is clearly a quantum channel and also verifies $T^*(\sigma_A \otimes \sigma_B) = \sigma_A \otimes \sigma_B$. Hence, in virtue of Lemma 5.4.1, we have

$$\| \tilde{O}_{AB} \|_{\mathbb{L}_1(\sigma_A \otimes \sigma_B)} \leq \| O_{AB} \|_{\mathbb{L}_1(\sigma_A \otimes \sigma_B)},$$

and, therefore,

$$\sup_{\|O_{AB}\|_{\mathbb{L}_1(\sigma_A \otimes \sigma_B)} \leq 1} \text{tr} \left[\tilde{O}_{AB} (\sigma_{AB} - \sigma_A \otimes \sigma_B) \right] \leq \sup_{\|\Omega_{AB}\|_{\mathbb{L}_1(\sigma_A \otimes \sigma_B)} \leq 1} \text{tr} [\Omega_{AB} (\sigma_{AB} - \sigma_A \otimes \sigma_B)].$$

In this last supremum over elements of 1-norm, we can undo the previous transformations in order to obtain again an ∞ -norm. First, we need to write the term in the supremum as a $\sigma_A \otimes \sigma_B$ -product of two terms:

$$\begin{aligned}
\text{tr} [\Omega_{AB} (\sigma_{AB} - \sigma_A \otimes \sigma_B)] &= \\
&= \text{tr} \left[(\sigma_A \otimes \sigma_B)^{1/2} (\sigma_A \otimes \sigma_B)^{-1/2} \sigma_{AB} (\sigma_A \otimes \sigma_B)^{-1/2} (\sigma_A \otimes \sigma_B)^{1/2} \Omega_{AB} \right] \\
&\quad - \text{tr} \left[(\sigma_A \otimes \sigma_B)^{1/2} \Omega_{AB} (\sigma_A \otimes \sigma_B)^{1/2} \right] \\
&= \left\langle \Omega_{AB}, (\sigma_A \otimes \sigma_B)^{-1/2} \sigma_{AB} (\sigma_A \otimes \sigma_B)^{-1/2} \right\rangle_{\sigma_A \otimes \sigma_B} - \langle \Omega_{AB}, \mathbb{1}_{AB} \rangle_{\sigma_A \otimes \sigma_B} \\
&= \left\langle \Omega_{AB}, (\sigma_A \otimes \sigma_B)^{-1/2} \sigma_{AB} (\sigma_A \otimes \sigma_B)^{-1/2} - \mathbb{1}_{AB} \right\rangle_{\sigma_A \otimes \sigma_B}.
\end{aligned}$$

Finally, using again the property of duality for the norms of $\mathbb{L}_1(\sigma_A \otimes \sigma_B)$ and $\mathbb{L}_\infty(\sigma_A \otimes \sigma_B)$, we have:

$$\begin{aligned}
& \sup_{\|\Omega_{AB}\|_{\mathbb{L}_1(\sigma_A \otimes \sigma_B)} \leq 1} \text{tr}[\Omega_{AB}(\sigma_{AB} - \sigma_A \otimes \sigma_B)] \\
&= \sup_{\|\Omega_{AB}\|_{\mathbb{L}_1(\sigma_A \otimes \sigma_B)} \leq 1} \left\langle \Omega_{AB}, (\sigma_A \otimes \sigma_B)^{-1/2} \sigma_{AB} (\sigma_A \otimes \sigma_B)^{-1/2} - \mathbb{1}_{AB} \right\rangle_{\sigma_A \otimes \sigma_B} \\
&= \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_{\mathbb{L}_\infty(\sigma_A \otimes \sigma_B)} \\
&= \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty,
\end{aligned}$$

where we have used again the fact that $\|\cdot\|_{\mathbb{L}_\infty(\rho_{AB})}$ coincides with the usual ∞ -norm for every $\rho_{AB} \in \mathcal{S}_{AB}$.

In conclusion,

$$\|\mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB}\|_\infty \leq \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty.$$

■

By putting together Step 5.1.1, Step 5.2.5, Step 5.3.2 and Step 5.4.2, we conclude the proof of Theorem 5.0.1.

5.5 IMPLICATIONS OF THIS RESULT

In the final section of this chapter, we briefly discuss some implications of the quantitative extension for the property of superadditivity proven above.

R Remark 5.5.1

This result constitutes an extension of the superadditivity property, since the term $H(\sigma_{AB})$ that appears in the statement of the main theorem vanishes when $\sigma_{AB} = \sigma_A \otimes \sigma_B$ and is small whenever $\sigma_{AB} \sim \sigma_A \otimes \sigma_B$. A trivial upper bound can be found with respect to the trace distance as follows,

$$\begin{aligned}
& \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty = \\
&= \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} (\sigma_{AB} - \sigma_A \otimes \sigma_B) \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \right\|_\infty \\
&\leq \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} (\sigma_{AB} - \sigma_A \otimes \sigma_B) \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \right\|_1 \\
&\leq \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \right\|_\infty \|\sigma_{AB} - \sigma_A \otimes \sigma_B\|_1 \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \right\|_\infty \\
&\leq \sigma_{\min}^{-2} \|\sigma_{AB} - \sigma_A \otimes \sigma_B\|_1,
\end{aligned}$$

where σ_{\min} is the minimum eigenvalue of σ_{AB} .

R Remark 5.5.2

The term $\|H(\sigma_{AB})\|_\infty$ is also closely related to certain forms of *decay of correlations* of states that have already appeared in quantum many body systems, such as *LTQO (Local Topological Quantum Order)* [MP13], or the concept of *local indistinguishability* as a strengthened form of *weak clustering* in [KB16].

Let us assume that $\|H(\sigma_{AB})\|_\infty \leq \lambda(\ell)$ for a certain small scalar $\lambda(\ell)$ that decays sufficiently fast as a function of the distance ℓ between regions A and B in a many body system, and denote by $\langle f \rangle_\varphi$ the expected value of an observable $f \in \mathcal{A}_{AB}$ with respect to a state φ (usually the ground or thermal state of the system). Then, for every observable of the form $O_A \otimes O_B \geq 0$, if we denote the reduced density matrix on AB of φ by σ_{AB} , the previous condition can be rewritten as

$$\left| \langle O_A O_B \rangle_\varphi - \langle O_A \rangle_\varphi \langle O_B \rangle_\varphi \right| \leq \lambda \langle O_A \rangle_\varphi \langle O_B \rangle_\varphi.$$

One can now compare this expression with the definition of decay of correlations

$$\left| \langle O_A O_B \rangle_\varphi - \langle O_A \rangle_\varphi \langle O_B \rangle_\varphi \right| \leq \lambda(\ell) \|O_A\|_\infty \|O_B\|_\infty,$$

or LTQO

$$\left| \langle O_A O_B \rangle_\varphi - \langle O_A \rangle_\varphi \langle O_B \rangle_\varphi \right| \leq \lambda(\ell) \langle O_A \rangle_\varphi \|O_B\|_\infty.$$

In conclusion, in this chapter we have proven an extension of the property of superadditivity of the quantum relative entropy for general states, a result that constitutes an improvement to the usual lower bound for the relative entropy of two bipartite states, given by the property of monotonicity, in terms of the relative entropies in the two constituent spaces, whenever the second state is near to be a tensor product.

Therefore, it might be relevant for situations where this property is expected to hold, such as quantum many body systems, in which it is likely that the Gibbs state satisfies this property in spatially separated systems.

In [KB16], Kastoryano and Brandao proved, for certain Gibbs samplers, the existence of a positive spectral gap for the dissipative dynamics, via a quasi-factorization result of the variance. This provides a bound for the mixing time of the evolution of the semigroup that drives the system to thermalization which is polynomial in the system size.

Following the same steps, we can use the main result of this chapter to obtain a result of quasi-factorization of the relative entropy in quantum many body systems (see Section 7.3), which will allow us to prove, under some conditions of decay of correlations on the Gibbs state, the existence of a positive log-Sobolev constant for the heat-bath dynamics in 1D, obtaining an exponential improvement in the bound for the mixing time obtained in [KB16] in some specific cases (see Chapter 10).



6. QUANTUM CONDITIONAL RELATIVE ENTROPY

In this chapter, we present an axiomatic definition of *conditional relative entropy*. Our aim is to introduce a concept that, given the value of the distinguishability between two states in a certain subsystem, quantifies their distinguishability in the whole space. More specifically, for two states in a bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the conditional relative entropy in A should provide the effect of the relative entropy of those states in the global space conditioned to the value of their relative entropy in B , extending the classical definition of *conditional entropy* of a function (see Definition 3.5.1).

Providing axiomatic definitions or presenting axiomatic characterizations for information theory quantities is a natural problem in quantum information theory. In particular, one can find in the literature several characterizations for the relative entropy, or related quantities (see [AD15], [Csi08], [Mül+13], [Pet92], [Rén03], among others).

However, for our definition, we rely on the recent work [WGE17], where the authors present an axiomatic characterization of the relative entropy, using strongly a previous result of Matsumoto [Mat10] (see Theorem 4.1.13). Indeed, as we saw in the former, they show that the properties of *continuity* (with respect to the first state), *monotonicity*, *additivity* and *superadditivity* characterize the relative entropy. This proof relies on two facts: The fact that the properties of continuity, additivity and superadditivity imply the so-called lower asymptotic semicontinuity, and the aforementioned result [Mat10], where it was proven that any function satisfying monotonicity, additivity and lower asymptotic semicontinuity is a multiple of the relative entropy.

The outline of the current chapter is the following: In the first section, we introduce the concept of conditional relative entropy from a collection of properties it should satisfy. In Section 6.2, we find the expression for the unique quantity that satisfies these properties. In the next section, we weaken the previous definition and introduce the notion of *conditional relative entropy by expectations*. Subsequently, in Section 6.4, we compare both definitions, providing examples for which they coincide, and in Section 6.5 we show that both of them extend the classical definition of conditional entropy. Finally, in the last section of the chapter, we introduce an alternative definition of conditional relative entropy that will allow us to obtain bounds on the log-Sobolev constant for the Davies dynamics in Chapter 11 via the strong quasi-factorization result presented in Chapter 8.

Beautiful view of Sydney in winter, where I attended *XIII Conference on the Theory of Quantum Computation, Communication and Cryptography (TQC 2017)*, in July 2017.

6.1 CONDITIONAL RELATIVE ENTROPY

We present the concept of quantum relative entropy as a function of two states verifying a collection of desired properties. The property of monotonicity is not expected to hold for this concept, as it does for the usual relative entropy, since the effect of A and B is not considered equally in the conditional relative entropy in A , so an arbitrary quantum channel (for instance, the partial trace in B) is not expected to decrease this quantity. For the same reason, additivity and superadditivity are neither expected to be true; however, for them, a property with the same spirit can be considered.

Definition 6.1.1 — CONDITIONAL RELATIVE ENTROPY, (CLP18a).

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. We define a *conditional relative entropy* in A as a function

$$D_A(\cdot || \cdot) : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \rightarrow \mathbb{R}_0^+$$

verifying the following properties for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$:

1. **Continuity:** The map $\rho_{AB} \mapsto D_A(\rho_{AB} || \sigma_{AB})$ is continuous.
2. **Non-negativity:** $D_A(\rho_{AB} || \sigma_{AB}) \geq 0$ and
 - (2.1) $D_A(\rho_{AB} || \sigma_{AB}) = 0$ if, and only if, $\rho_{AB} = \mathbb{E}_A^*(\rho_{AB})$,
where $\mathbb{E}_A^*(\cdot)$ is the heat-bath conditional expectation introduced in Section 4.3,
and it is given by

$$\mathbb{E}_A^*(\rho_{AB}) = \sigma_{AB}^{1/2} \sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \sigma_{AB}^{1/2}.$$

3. **Semi-superadditivity:** $D_A(\rho_{AB} || \sigma_A \otimes \sigma_B) \geq D(\rho_A || \sigma_A)$ and
 - (3.1) **Semi-additivity:** if $\rho_{AB} = \rho_A \otimes \rho_B$, $D_A(\rho_A \otimes \rho_B || \sigma_A \otimes \sigma_B) = D(\rho_A || \sigma_A)$.
4. **Semi-monotonicity:** For every quantum channel $\mathcal{T} : \mathcal{S}_{AB} \rightarrow \mathcal{S}_{AB}$, the following inequality holds:

$$\begin{aligned} D_A(\mathcal{T}(\rho_{AB}) || \mathcal{T}(\sigma_{AB})) + D_B((\text{tr}_A \circ \mathcal{T})(\rho_{AB}) || (\text{tr}_A \circ \mathcal{T})(\sigma_{AB})) \\ \leq D_A(\rho_{AB} || \sigma_{AB}) + D_B(\text{tr}_A(\rho_{AB}) || \text{tr}_A(\sigma_{AB})), \end{aligned}$$

where $D_B(\rho_{AB} || \sigma_{AB})$ is the conditional relative entropy in B .

Let us recall that $\mathbb{E}_A^*(\cdot)$ coincides with the Petz recovery map for the partial trace composed with the partial trace. Hence, property (2.1) can be interpreted in some sense as a recovery condition.

Remark 6.1.2

Property (3.1) yields the fact that if we consider states with support in A , we recover the usual definition of relative entropy, i.e.,

$$D_A(\rho_A \otimes \mathbb{1}_B || \sigma_A \otimes \mathbb{1}_B) = D(\rho_A || \sigma_A)$$

In general, if \mathcal{T} is a quantum channel, note that the following holds,

$$D_A((\text{tr}_B \circ \mathcal{T})(\rho_{AB}) || (\text{tr}_B \circ \mathcal{T})(\sigma_{AB})) = D((\text{tr}_B \circ \mathcal{T})(\rho_{AB}) || (\text{tr}_B \circ \mathcal{T})(\sigma_{AB})).$$

since $(\text{tr}_B \circ \mathcal{T})(\rho_{AB})$ and $(\text{tr}_B \circ \mathcal{T})(\sigma_{AB})$ have support in A .

R Remark 6.1.3

Note that, by virtue of property (2.1), if $\rho_{AB} = \sigma_{AB}$, in particular $\rho_{AB} = \mathbb{E}_A^*(\rho_{AB})$ and, thus, $D_A(\rho_{AB}||\sigma_{AB})=0$. However, the converse implication is false in general (differently from the case of the relative entropy). Indeed, both implications cannot hold simultaneously, since that would be incompatible with property (3.1).

Let us define

$$D_{A,B}^+(\rho_{AB}||\sigma_{AB}) := D_A(\rho_{AB}||\sigma_{AB}) + D_B(\rho_{AB}||\sigma_{AB}).$$

Then, we can prove a couple of properties for $D_{A,B}^+$ that yield some relation of this concept with the usual relative entropy.

Proposition 6.1.4 Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. $D_{A,B}^+$ satisfies the following properties:

1. **Additivity:** $D_{A,B}^+(\rho_A \otimes \rho_B || \sigma_A \otimes \sigma_B) = D(\rho_A || \sigma_A) + D(\rho_B || \sigma_B)$.
2. **Superadditivity:** $D_{A,B}^+(\rho_{AB} || \sigma_A \otimes \sigma_B) \geq D(\rho_A || \sigma_A) + D(\rho_B || \sigma_B)$.

Proof. • (1) follows from property (3.1) in the definition of conditional relative entropy.
• (2) is obtained from property (3) in the definition of conditional relative entropy. ■

Properties (1), (2), (3) and (3.1) are necessary so that the conditional relative entropy extends the relative entropy. The names of properties (3) and (3.1) come from the fact seen above that $D_{A,B}^+$ actually satisfies the properties of additivity and superadditivity.

Note from the previous definition that the main difference between the relative entropy and $D_{A,B}^+$ lies in the fact that the latter lacks the property of monotonicity. Indeed, as mentioned above, since $D_{A,B}^+$ verifies the properties of continuity, additivity and superadditivity, we know that it cannot verify the property of monotonicity (i.e., data processing for every quantum channel), as it would imply that it is a multiple of the relative entropy [WGE17]. This motivates the appearance of the property of “semi-monotonicity”.

To justify the name for that property, let us comment a bit on every term of the inequality that defines it. Comparing the first term of both sides of the inequality, we can find a data processing inequality for D_A . Such inequality cannot hold in general, since, for the conditional relative entropy in A , a quantum channel with support in B is not expected to decrease this quantity. This fact justifies the presence of the second term on both sides of the inequality, to compensate the non-decreasing effect of the “ B -part” of a channel in the conditional relative entropy in A . by adding the conditional relative entropy of this “ B -part” of the channel in B , where we know that the decreasing effect actually holds.

In the following section, we will show that there exists a unique expression for the conditional relative entropy satisfying the properties of Definition 6.1.1.

6.2 A FORMULA FOR THE CONDITIONAL RELATIVE ENTROPY

The main result of this section is a characterization of the conditional relative entropy.

Theorem 6.2.1 — AXIOMATIC CHARACTERIZATION OF THE CRE, (CLP18a).

Let $D_A(\cdot||\cdot)$ be a conditional relative entropy, according to Definition 6.1.1. Then, $D_A(\cdot||\cdot)$ is explicitly given by

$$D_A(\rho_{AB}||\sigma_{AB}) = D(\rho_{AB}||\sigma_{AB}) - D(\rho_B||\sigma_B),$$

for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$.

Proof. Let us first prove that the quantity D_A fulfills all the conditions in Definition 6.1.1. Let us recall that we need prove the following properties:

1. *The map $\rho_{AB} \mapsto D_A(\rho_{AB}||\sigma_{AB})$ is continuous.*

It is clear that $D(\rho_{AB}||\sigma_{AB})$ is continuous in ρ_{AB} , so $D(\rho_B||\sigma_B)$ also is. Hence, their difference is also continuous.

2. *$D_A(\rho_{AB}||\sigma_{AB}) \geq 0$ and*

(2.1) $D_A(\rho_{AB}||\sigma_{AB})=0$ if, and only if, $\rho_{AB} = \mathbb{E}_A^(\rho_{AB})$.*

Note that (2) is the monotonicity of the relative entropy (property (3) of Proposition 4.1.5) for the channel tr_A , and Property (2.1) was proven in [Pet03].

3. *$D_A(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A)$ and*

(3.1) if $\rho_{AB} = \rho_A \otimes \rho_B$, $D_A(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A)$.

In (3), using the superadditivity of the relative entropy, we get

$$D_A(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B) - D(\rho_B||\sigma_B) = D(\rho_A||\sigma_A).$$

Moreover, for (3.1), we have equality in the previous inequality:

$$D_A(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B) - D(\rho_B||\sigma_B) = D(\rho_A||\sigma_A).$$

4. *For every quantum channel \mathcal{T} , the following holds:*

$$\begin{aligned} D_A(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB})) + D_B((\text{tr}_A \circ \mathcal{T})(\rho_{AB})||(\text{tr}_A \circ \mathcal{T})(\sigma_{AB})) \\ \leq D_A(\rho_{AB}||\sigma_{AB}) + D_B(\text{tr}_A(\rho_{AB})||\text{tr}_A(\sigma_{AB})), \end{aligned}$$

where $D_B(\rho_{AB}||\sigma_{AB})$ is the conditional relative entropy in B .

The first term in the LHS is expressed as:

$$D_A(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB})) = D(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB})) - D_B((\text{tr}_A \circ \mathcal{T})(\rho_{AB})||(\text{tr}_A \circ \mathcal{T})(\sigma_{AB})).$$

Hence, the LHS of the statement of the proposition is actually given by

$$D_A(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB})) + D_B((\text{tr}_A \circ \mathcal{T})(\rho_{AB})||(\text{tr}_A \circ \mathcal{T})(\sigma_{AB})) = D(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB})).$$

Now, for the first term in the RHS, we have

$$D_A(\rho_{AB}||\sigma_{AB}) = D(\rho_{AB}||\sigma_{AB}) - D(\rho_B||\sigma_B).$$

Thus, the RHS can be rewritten as

$$\begin{aligned} D_A(\rho_{AB}||\sigma_{AB}) + D_B(\text{tr}_A(\rho_{AB})||\text{tr}_A(\sigma_{AB})) &= D(\rho_{AB}||\sigma_{AB}) + D(\rho_B||\sigma_B) \\ &= D(\rho_{AB}||\sigma_{AB}), \end{aligned}$$

where in the first line we have used Remark 6.1.

In conclusion, the statement of the property is equivalent to the following inequality:

$$D(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB})) \leq D(\rho_{AB}||\sigma_{AB}),$$

which holds for every quantum channel \mathcal{T} (property of monotonicity in Proposition 4.1.5).

Once we have proven that this definition of D_A is indeed a conditional relative entropy according to Definition 6.1.1, we can move forward to the proof of the converse implication.

Let us define $f : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \rightarrow \mathbb{R}_0^+$ by:

$$f(\rho_{AB}, \sigma_{AB}) = D_A(\rho_{AB} || \sigma_{AB}) + D(\rho_B || \sigma_B),$$

where D_A is a conditional relative entropy (and, hence, satisfies the properties of Definition 6.1.1). The aim of this part of the proof is to see that

$$f(\rho_{AB}, \sigma_{AB}) = D(\rho_{AB} || \sigma_{AB}).$$

In virtue of the characterization for the relative entropy shown in Theorem 4.1.13, we just need to prove that f satisfies the following properties for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$:

1. **Continuity:** $\rho_{AB} \mapsto f(\rho_{AB}, \sigma_{AB})$ is continuous.

It is a direct consequence of property (1) in Definition 6.1.1.

2. **Additivity:** $f(\rho_A \otimes \rho_B, \sigma_A \otimes \sigma_B) = f(\rho_A, \sigma_A) + f(\rho_B, \sigma_B)$.

This follows from property (3.1) in Definition 6.1.1.

3. **Superadditivity:** $f(\rho_{AB}, \sigma_A \otimes \sigma_B) \geq f(\rho_A, \sigma_A) + f(\rho_B, \sigma_B)$.

It is straightforward from property (3) in Definition 6.1.1.

4. **Monotonicity:** For every quantum channel \mathcal{T} ,

$$f(\mathcal{T}(\rho_{AB}), \mathcal{T}(\sigma_{AB})) \leq f(\rho_{AB}, \sigma_{AB}).$$

The second term in the definition of f can be rewritten as:

$$\begin{aligned} D(\rho_B || \sigma_B) &= D_B(\rho_B || \sigma_B) \\ &= D_B(\text{tr}_A[\rho_{AB}] || \text{tr}_A[\sigma_{AB}]), \end{aligned}$$

where we have used Remark 6.1 in the second line. Thus, we can write the property of monotonicity of f as:

$$\begin{aligned} D_A(\mathcal{T}(\rho_{AB}) || \mathcal{T}(\sigma_{AB})) + D_B((\text{tr}_A \circ \mathcal{T})(\rho_{AB}) || (\text{tr}_A \circ \mathcal{T})(\sigma_{AB})) \\ \leq D_A(\rho_{AB} || \sigma_{AB}) + D(\rho_B || \sigma_B), \end{aligned}$$

and this property holds by assumption, because of the property of semi-monotonicity. Hence, recalling Theorem 4.1.13, we can deduce that

$$f(\rho_{AB}, \sigma_{AB}) \leq D(\rho_{AB} || \sigma_{AB}).$$

Moreover, if we take $\rho_{AB} = \rho_A \otimes \rho_B$ and $\sigma_{AB} = \sigma_A \otimes \sigma_B$, we have

$$\begin{aligned} f(\rho_A \otimes \rho_B, \sigma_A \otimes \sigma_B) &= D_A(\rho_A \otimes \rho_B || \sigma_A \otimes \sigma_B) + D(\rho_B || \sigma_B) \\ &= D(\rho_A || \sigma_A) + D(\rho_B || \sigma_B) \\ &= D(\rho_A \otimes \rho_B || \sigma_A \otimes \sigma_B), \end{aligned}$$

from which we can conclude

$$f(\rho_{AB}, \sigma_{AB}) = D(\rho_{AB} || \sigma_{AB}).$$

This fact immediately yields the statement of the theorem. ■

R Remark 6.2.2

Throughout the whole paper we are assuming that all the states considered are full-rank, and, thus, their relative entropy is finite. Hence, the conditional relative entropy, which we have just seen that can be expressed as a difference of relative entropies, is the difference of two finite quantities, so it is always well-defined.

The formula obtained in this subsection for the conditional relative entropy allows us to give an operational interpretation to this quantity. In the context of thermodynamics and cost of quantum processes, in [FR18], the authors introduced the concept of *coherent relative entropy* to give a measure of the amount of information forgotten by a logical process, conditioned to the output of the process, and relative to certain weights encoded in an operator. In thermodynamics, this quantity can be seen as the work cost of a certain quantum process (some applications and interesting properties of this quantity have appeared in [FBB18]).

Our conditional relative entropy coincides with the coherent relative entropy when the process considered is a partial trace and taking the i.i.d. limit, which allows us to think that the relative entropy might be of use in a wider range of physical and information-theoretic situations.

6.3 CONDITIONAL RELATIVE ENTROPY BY EXPECTATIONS

In the previous section we have shown that there exists a unique conditional relative entropy fulfilling all properties from Definition 6.1.1. However, whereas the properties of continuity, non-negativity, semi-additivity and semi-superadditivity are expected to hold for such concept of conditional relative entropy, the property of semi-monotonicity, although justified below the definition, may seem less natural.

One could then think of modifying, or just removing this property from the definition. That would leave space for more possible examples of this new *modified conditional relative entropy*. The purpose of this subsection is, indeed, to introduce an example of a modified conditional relative entropy that lacks the property of semi-monotonicity.

One quantity widely used in quantum information theory is the relative entropy between a state and its recovery by means of the Petz recovery map for the partial trace. Indeed, it is known that there are cases where this quantity coincides with the aforementioned conditional relative entropy (this will be further discussed in Section 6.4). Hence, it is also natural to study this quantity as a possible modified conditional relative entropy.

Definition 6.3.1 — CONDITIONAL RELATIVE ENTROPY BY EXPECTATIONS, (CLP18a).

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a composite Hilbert space and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. Let \mathbb{E}_A^* be the adjoint of the heat-bath conditional expectation introduced in Section 4.3. We define the *conditional relative entropy by expectations* of ρ_{AB} and σ_{AB} in A by:

$$D_A^E(\rho_{AB} || \sigma_{AB}) := D(\rho_{AB} || \mathbb{E}_A^*(\rho_{AB})).$$

Let us check now that this quantity is indeed a modified conditional relative entropy in the sense above, by proving that it fulfills all the properties of the axiomatic definition of conditional relative entropy except for the semi-monotonicity.

Proposition 6.3.2 Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. The following properties hold for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$:

1. The map $\rho_{AB} \mapsto D_A^E(\rho_{AB} || \sigma_{AB})$ is continuous.
2. $D_A^E(\rho_{AB} || \sigma_{AB}) \geq 0$ and
 - (2.1) $D_A^E(\rho_{AB} || \sigma_{AB}) = 0$ if, and only if, $\rho_{AB} = \mathbb{E}_A^*(\rho_{AB})$.
3. $D_A^E(\rho_{AB} || \sigma_A \otimes \sigma_B) \geq D(\rho_A || \sigma_A)$ and
 - (3.1) if $\rho_{AB} = \rho_A \otimes \rho_B$, $D_A^E(\rho_A \otimes \rho_B || \sigma_A \otimes \sigma_B) = D(\rho_A || \sigma_A)$.

Proof. • (1) is due to the facts that $\mathbb{E}_A^*(\rho_{AB})$ is linear in ρ_{AB} and the relative entropy is continuous.

- Property (2) comes from the fact that the conditional relative entropy by expectations is in particular a relative entropy of density matrices.
- (2.1) is a consequence of the fact that the relative entropy of two states vanishes if, and only if, they coincide.
- For (3), observe that if $\sigma_{AB} = \sigma_A \otimes \sigma_B$,

$$\begin{aligned} \mathbb{E}_A^*(\rho_{AB}) &= \sigma_A^{1/2} \otimes \sigma_B^{1/2} \sigma_B^{-1/2} \rho_{AB} \sigma_B^{-1/2} \sigma_A^{1/2} \otimes \sigma_B^{1/2} \\ &= \sigma_A \otimes \rho_B. \end{aligned}$$

Hence,

$$\begin{aligned} D_A^E(\rho_{AB} || \sigma_A \otimes \sigma_B) &= D(\rho_{AB} || \sigma_A \otimes \rho_B) \\ &= D(\rho_{AB} || \rho_A \otimes \rho_B) + D(\rho_A || \sigma_A) \\ &\geq D(\rho_A || \sigma_A), \end{aligned}$$

where we have used the non-negativity of the relative entropy.

- In (3.1), if both ρ_{AB} and σ_{AB} are tensor products, we have equality in the previous inequality:

$$\begin{aligned} D_A^E(\rho_A \otimes \rho_B || \sigma_A \otimes \sigma_B) &= D(\rho_A \otimes \rho_B || \sigma_A \otimes \rho_B) \\ &= D(\rho_A || \sigma_A), \end{aligned}$$

since $D(\rho_A \otimes \rho_B || \sigma_A \otimes \rho_B) = D(\rho_A || \sigma_A) + D(\rho_B || \rho_B)$ and the second term is zero. ■

6.4 COMPARISON OF DEFINITIONS

Once we have presented both definitions for conditional relative entropy and conditional relative entropy by expectations, respectively, it is a natural question whether they coincide in general and, if not, characterize the states for which they do. Let us consider ρ_{AB} and σ_{AB} bipartite density matrices in $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and study different cases.

Case 1: ρ_B, σ_{AB} and σ_B commute.

We first assume that $[\rho_B, \sigma_{AB}] = [\rho_B, \sigma_B] = [\sigma_B, \sigma_{AB}] = 0$. Then, we can rewrite both definitions of conditional relative entropies as:

$$\begin{aligned} D_A(\rho_{AB} || \sigma_{AB}) &= D(\rho_{AB} || \sigma_{AB}) - D(\rho_B || \sigma_B) \\ &= \text{tr}[\rho_{AB}(\log \rho_{AB} - \log \sigma_{AB} \rho_B \sigma_B^{-1})] \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} D_A^E(\rho_{AB} || \sigma_{AB}) &= D(\rho_{AB} || \mathbb{E}_A^*(\rho_{AB})) \\ &= \text{tr}[\rho_{AB}(\log \rho_{AB} - \log \sigma_{AB} \rho_B \sigma_B^{-1})], \end{aligned} \tag{6.2}$$

so we can see that they coincide.

Case 2: σ_{AB} has the splitting property.

Suppose that $\sigma_{AB} = \sigma_A \otimes \sigma_B$. Then, for the conditional relative entropy, we have

$$D_A(\rho_{AB} || \sigma_A \otimes \sigma_B) = D(\rho_{AB} || \sigma_A \otimes \sigma_B) - D(\rho_B || \sigma_B) \quad (6.3)$$

$$= D(\rho_{AB} || \rho_A \otimes \rho_B) + D(\rho_A \otimes \rho_B || \sigma_A \otimes \sigma_B) - D(\rho_B || \sigma_B) \quad (6.4)$$

$$= I_\rho(A : B) + D(\rho_A || \sigma_A) + D(\rho_B || \sigma_B) - D(\rho_B || \sigma_B) \quad (6.5)$$

$$= I_\rho(A : B) + D(\rho_A || \sigma_A).$$

Furthermore, the heat-bath conditional expectation takes the value $\mathbb{E}_A^*(\rho_{AB}) = \sigma_A \otimes \rho_B$. Thus, the conditional relative entropy by expectations in this case is given by

$$D_A^E(\rho_{AB} || \sigma_{AB}) = D(\rho_{AB} || \sigma_A \otimes \rho_B) \quad (6.6)$$

$$= D(\rho_{AB} || \rho_A \otimes \rho_B) + D(\rho_A \otimes \rho_B || \sigma_A \otimes \rho_B) \quad (6.7)$$

$$= I_\rho(A : B) + D(\rho_A || \sigma_A).$$

Therefore,

$$D_A(\rho_{AB} || \sigma_A \otimes \sigma_B) = D_A^E(\rho_{AB} || \sigma_A \otimes \sigma_B).$$

Case 3: $D_A(\rho_{AB} || \sigma_{AB}) = 0$ or $D_A^E(\rho_{AB} || \sigma_{AB}) = 0$.

On the one hand, for the conditional relative entropy by expectations, as it is in particular a relative entropy between two states it is clear that (Proposition 4.1.5) the following holds:

$$D_A^E(\rho_{AB} || \sigma_{AB}) = 0 \Leftrightarrow \rho_{AB} = \mathbb{E}_A^*(\rho_{AB}).$$

On the other hand, for the conditional relative entropy, the situation

$$D_A(\rho_{AB} || \sigma_{AB}) = 0 \Leftrightarrow D(\rho_{AB} || \sigma_{AB}) = D(\rho_B || \sigma_B)$$

was addressed and characterized by Petz in [Pet03]. In general, if \mathcal{H} and \mathcal{K} are two Hilbert spaces, we have already recalled in the Introduction (see Chapter 1) and will explain in further detail in Part IV that there is equality in the data processing inequality for a quantum channel \mathcal{T} ([Uhl77][Lin75])

$$D(\rho || \sigma) \geq D(\mathcal{T}(\rho) || \mathcal{T}(\sigma)), \quad (6.8)$$

if, and only if, both ρ and σ can be recovered in the following way

$$\widehat{\mathcal{T}}\mathcal{T}(\rho) = \rho, \quad \widehat{\mathcal{T}}\mathcal{T}(\sigma) = \sigma,$$

where $\widehat{\mathcal{T}}$ can be explicitly given by:

$$\widehat{\mathcal{T}}\eta = \sigma^{1/2} \mathcal{T}^* \left((\mathcal{T}\sigma)^{-1/2} \eta (\mathcal{T}\sigma)^{-1/2} \right) \sigma^{1/2},$$

for a state $\eta \in \mathcal{S}(\mathcal{K})$. Note from the expression of $\widehat{\mathcal{T}}$ that $\widehat{\mathcal{T}}\mathcal{T}(\sigma) = \sigma$ always holds.

For the particular case of the partial trace, this problem was also addressed in [Hay+04], and based on the fact that having equality in Equation (6.8) for this channel is equivalent to having equality in the strong subadditivity of the relative entropy (Proposition 4.1.7), they provided the decomposition shown in Theorem 4.7.2.

In what concerns equality in (6.8) for the partial trace, Petz' result reads as:

$$D_A(\rho_{AB} || \sigma_{AB}) = 0 \Leftrightarrow \rho_{AB} = \sigma_{AB}^{1/2} \sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \sigma_{AB}^{1/2} = \mathbb{E}_A^*(\rho_{AB}),$$

where we recall that $\mathbb{E}_A^*(\rho_{AB})$ is exactly the Petz recovery map for the partial trace tr_A .

Therefore, the kernels of both definitions of conditional relative entropies coincide, i.e., both vanish under the same conditions.

Case 4: General case.

We have seen that both definitions mentioned above coincide, at least, when they are null, σ_{AB} is a tensor product, or $[\rho_B, \sigma_{AB}] = [\rho_B, \sigma_B] = [\sigma_B, \sigma_{AB}] = 0$. In general, as far as we know, the problem of characterizing for which states ρ_{AB}, σ_{AB} both definitions coincide is still an open question.

Another natural question that arises in this context is whether one definition could be always greater or equal than the other, i.e., whether the following inequality

$$D_A(\rho_{AB}||\sigma_{AB}) \stackrel{?}{\geq} D_A^E(\rho_{AB}||\sigma_{AB}) \quad (6.9)$$

or the reverse one hold for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The left-hand side of this inequality has been widely studied in a series of recent papers ([FR15], [BLW15], [DW16], [Jun+18], [SBT17], among other results), where the authors provide several lower and upper bounds for our conditional relative entropy by differences. These results will be further discussed in Part IV.

Nevertheless, inequality (6.9) is already known to be false in general. Let us consider a tripartite Hilbert space $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and compare both definitions of conditional relative entropy in C . Consider $\rho_{ABC} \in \mathcal{S}_{ABC}$ and suppose that $\sigma_{ABC} = \mathbb{1}_A \otimes \rho_{BC}$. Then,

$$\begin{aligned} D_C(\rho_{ABC}||\sigma_{ABC}) &= D(\rho_{ABC}||\sigma_{ABC}) - D(\rho_{AB}||\sigma_{AB}) \\ &= D(\rho_{ABC}||\rho_{BC}) - D(\rho_{AB}||\rho_B) \\ &= -S[\rho_{ABC}] + S[\rho_{BC}] + S[\rho_{AB}] - S[\rho_B] \\ &= I_\rho(A : BC) - I_\rho(A : B) \\ &= I_\rho(A : C|B), \end{aligned}$$

where this last term is called *conditonal mutual information*, and

$$\begin{aligned} D_C^E(\rho_{ABC}||\sigma_{ABC}) &= D(\rho_{ABC}||\mathbb{E}_C^*(\rho_{ABC})) \\ &= D(\rho_{ABC}||\rho_{BC}^{1/2} \rho_B^{-1/2} \rho_{AB} \rho_B^{-1/2} \rho_{BC}^{1/2}). \end{aligned}$$

Hence, inequality (6.9) in the particular case $\sigma_{ABC} = \mathbb{1}_A \otimes \rho_{BC}$ can be rewritten as

$$I_\rho(A : C|B) \stackrel{?}{\geq} D(\rho_{ABC}||\rho_{BC}^{1/2} \rho_B^{-1/2} \rho_{AB} \rho_B^{-1/2} \rho_{BC}^{1/2}). \quad (6.10)$$

This problem was addressed in [Bra+15b], where they considered these two quantities and plotted one against the other for 10.000 randomly chosen pure states of dimension $2 \times 2 \times 2$. They showed that even though in most of the cases the conditional mutual information is strictly greater than the conditional relative entropy by expectations, there are cases in which the reverse inequality holds. Similar numerical results had also been obtained in [LW18].

In the recent paper [FF18], the authors studied the following inequality:

$$I_\rho(A : C|B) \stackrel{?}{\geq} \min_{\Lambda: B \rightarrow BC} D(\rho_{ABC}||(\mathbb{1}_A \otimes \Lambda)(\rho_{AB})). \quad (6.11)$$

They tested it on 2.000 randomly chosen pure states of dimension $2 \times 2 \times 2$ and showed that there are states for which inequality (6.11) is violated. For these states, in particular, inequality (6.10) is also violated, since

$$I_\rho(A : C|B) < \min_{\Lambda: B \rightarrow BC} D(\rho_{ABC}||(\mathbb{1}_A \otimes \Lambda)(\rho_{AB})) \leq D(\rho_{ABC}||\rho_{BC}^{1/2} \rho_B^{-1/2} \rho_{AB} \rho_B^{-1/2} \rho_{BC}^{1/2}).$$

Subsequently, they also presented an explicit counterexample for inequality (6.11).

R Remark 6.4.1

A natural question in this context is whether one can recover the conditional entropy of ρ_{AB} when one considers $\sigma_{AB} = \mathbb{1}_{AB}$ in the conditional relative entropy, analogously to what happens for the von Neumann entropy of ρ_{AB} , which is recovered from the relative entropy of ρ_{AB} and σ_{AB} when $\sigma_{AB} = \mathbb{1}_{AB}$.

More specifically, given a bipartite Hilbert space and ρ_{AB} a state on it, for the conditional relative entropy in A of ρ_{AB} and $\mathbb{1}_{AB}$ (and thus the conditional relative entropy by expectations, since they coincide in this case) we have:

$$\begin{aligned} D_A(\rho_{AB} || \mathbb{1}_{AB}) &= D(\rho_{AB} || \mathbb{1}_{AB}) - D(\rho_B || d_A \mathbb{1}_B) \\ &= -S(\rho_{AB}) + S(\rho_B) + \text{tr}[\rho_B \log d_A] \\ &= -S(A|B)_\rho + \log d_A. \end{aligned}$$

Hence, from both definitions we can recover the conditional entropy of ρ in A plus an additive factor with the logarithm of the dimension of \mathcal{H}_A , due to the fact that both definitions of conditional relative entropies were provided for states, instead of observables. If we compute both conditional relative entropies of ρ_{AB} and $\mathbb{1}_{AB}/d_{AB}$ (because now they are both states), then we recover the conditional entropy of ρ_{AB} in both situations.

6.5 RELATION WITH THE CLASSICAL CASE

In this section, we will prove that both definitions presented above extend their classical analogue. Before that, we recall the classical definition for the entropy and the conditional entropy of a function, respectively, introduced in more detail in Chapter 3.

Let us recall that for a probability space $(\Omega, \mathcal{F}, \mu)$ and for every $f > 0$, the *entropy* of f is defined by

$$\text{Ent}_\mu(f) := \mu(f \log f) - \mu(f) \log \mu(f).$$

Moreover, given a sub- σ -algebra $\mathcal{G} \subseteq \mathcal{F}$, the *conditional entropy* of f in \mathcal{G} is given by

$$\text{Ent}_\mu(f | \mathcal{G}) := \mu(f \log f | \mathcal{G}) - \mu(f | \mathcal{G}) \log \mu(f | \mathcal{G}),$$

where $\mu(f | \mathcal{G})$ is given by

$$\int_G \mu(f | \mathcal{G}) d\mu = \int_G f d\mu \text{ for each } G \in \mathcal{G}.$$

Let us consider two measures ν and μ in (Ω, \mathcal{F}) . We define the *relative entropy* of ν with respect to μ by

$$H(\nu | \mu) := \begin{cases} \mu(f \log f) & \text{if } d\nu = f d\mu, f \log f \in L^1(\mu), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, we can relate it to the previous concept by

$$H(\nu | \mu) = \text{Ent}_\mu \left(\frac{d\nu}{d\mu} \right),$$

and analogously to the definition of conditional entropy, we could define the *conditional relative entropy* of ν with respect to μ in \mathcal{G} by:

$$H_{\mathcal{G}}(\nu | \mu) = \text{Ent}_\mu(f | \mathcal{G}),$$

for $f = \frac{dv}{d\mu}$.

Let us compare now this setting to the quantum case. We will prove that, when the states are classical, both the conditional relative entropy and the conditional relative entropy by expectations coincide with the measure of the classical conditional entropy. The measure is necessary due to the fact that the classical conditional entropy of a function is another function, whereas the conditional relative entropy of two states produces a scalar.

We first rewrite the classical conditional entropy as:

$$\begin{aligned} \text{Ent}_\mu(f | \mathcal{G}) &= \mu(f \log f | \mathcal{G}) - \mu(f | \mathcal{G}) \log \mu(f | \mathcal{G}) \\ &= \mu(f \log f | \mathcal{G}) - \mu(f \log \mu(f | \mathcal{G}) | \mathcal{G}) \\ &= \mu(f(\log f - \log \mu(f | \mathcal{G})) | \mathcal{G}). \end{aligned}$$

Now, since $\mu(\mu(\cdot | \mathcal{G})) = \mu(\cdot)$,

$$\mu(\text{Ent}_\mu(f | \mathcal{G})) = \mu(f(\log f - \log \mu(f | \mathcal{G}))) \quad (6.12)$$

Let us consider a bipartite Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and a classical state on it, i.e., of the form:

$$\rho_{AB} = \sum_{a,b} P_{AB}(a,b) |a\rangle\langle a|_A \otimes |b\rangle\langle b|_B.$$

Then, since the space of observables for each system is an abelian C^* -algebra, in virtue of Gelfand's theorem (see [Arv76], for instance) the composite system of observables can be expressed as

$$C(K) \otimes C(L) = C(K \times L),$$

where both K and L are compact spaces. A state in the composite system is a positive ρ of the dual of $C(K \times L)$, which by the Riesz-Markov theorem ([Rie09], [Mar38]) corresponds to a regular Borel measure on $K \times L$. Hence, we can identify a classical state ρ_{AB} with a regular measure μ .

Moreover, we obtain the corresponding reduced state in one of the components by projecting the measure of ρ_{AB} to that component, so the partial trace of the quantum setting can be interpreted as this operation in the classical setting (which is exactly the operation of the conditioning to a sub- σ -algebra in the definition of the classical conditional entropy). Thus, we identify $\text{tr}_A[\cdot]$ with $\mu(\cdot | \mathcal{F})$.

Let us also recall that in the quantum setting we are considering states and in the definition of classical entropy, observables. The transition from the Schrödinger picture to the Heisenberg picture can be made by means of the operator:

$$\Gamma_{\sigma_{AB}}^{-1}(\rho_{AB}) = \sigma_{AB}^{-1/2} \rho_{AB} \sigma_{AB}^{-1/2}$$

for a certain full-rank state σ_{AB} , which we also consider classical. In particular, ρ_{AB} and σ_{AB} commute, as well as their marginals. This operator will appear often in Chapter 12.

If we put all this together (a diagram of this identification can be seen in Figure 6.1) along with Equation (6.12), and identify the trace with respect to σ_{AB} with the measure μ , taking into account that $f = dv/d\mu$ is identified with $\Gamma_{\sigma_{AB}}^{-1}(\rho_{AB})$, we have:

$$\begin{aligned} \mu(\text{Ent}_\mu(f | \mathcal{G})) &= \mu(f(\log f - \log \mu(f | \mathcal{G}))) \\ &= \text{tr}[\sigma_{AB} \Gamma_{\sigma_{AB}}^{-1}(\rho_{AB}) (\log \Gamma_{\sigma_{AB}}^{-1}(\rho_{AB}) - \log \text{tr}_A[\Gamma_{\sigma_{AB}}^{-1}(\rho_{AB})])] \\ &= \text{tr}[\rho_{AB} (\log \rho_{AB} \sigma_{AB}^{-1} - \log \rho_B \sigma_B^{-1})] \\ &= \text{tr}[\rho_{AB} (\log \rho_{AB} - \log \sigma_{AB} - \log \rho_B + \log \sigma_B)] \\ &= D_A(\rho_{AB} || \sigma_{AB}) = D_A^E(\rho_{AB} || \sigma_{AB}), \end{aligned}$$

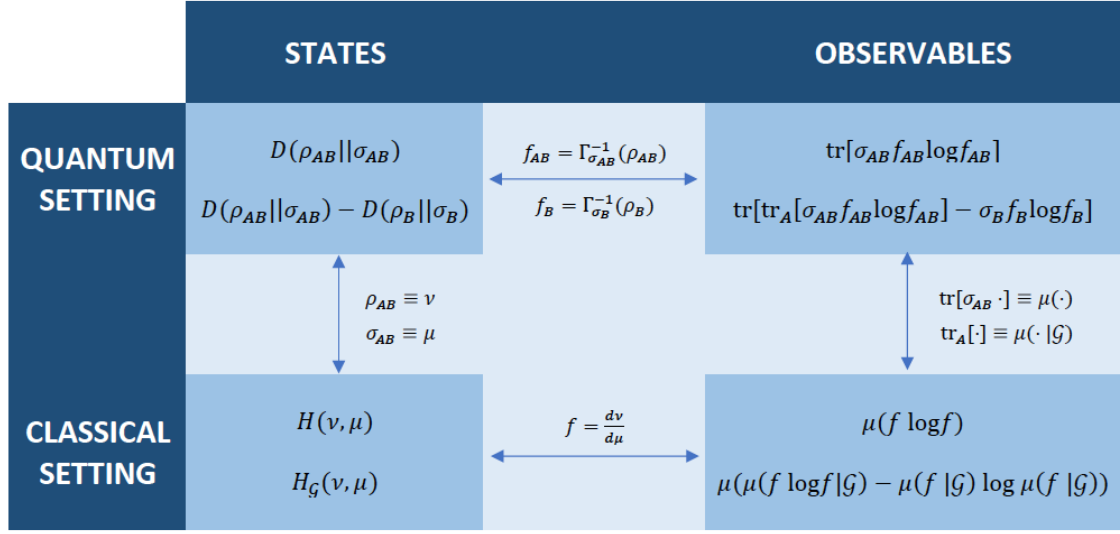


Figure 6.1: Identification between classical and quantum quantities when the states considered are classical.

where we have proven that both the quantum conditional relative entropy and the conditional relative entropy by expectations coincide with the measure of the classical conditional entropy.

6.6 GENERAL CONDITIONAL RELATIVE ENTROPY BY EXPECTATIONS

In the last section of this chapter, we introduce the *general conditional relative entropy by expectations*. Their name is due to the fact that they are conditional relative entropies that can be defined for any conditional expectation (see Section 4.3), and not just the heat-bath one, as in Section 6.3. In fact, since the heat-bath conditional expectation is not a true conditional expectation, according to Proposition 4.3.1, the conditional relative entropy by expectations does not constitute an example of the quantity introduced below.

Definition 6.6.1 — GENERAL CONDITIONAL RELATIVE ENTROPY BY EXPECTATIONS.

Let \mathcal{M} be a matrix algebra with matrix subalgebra \mathcal{N} . Let σ be a state of \mathcal{M} and consider a conditional expectation $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$. Then, we define the *general conditional relative entropy by expectations* for this conditional expectation as

$$D_{\mathcal{N}}^{\mathcal{E}}(\rho||\sigma) = D(\rho||\mathcal{E}^*(\rho)),$$

for every $\rho \in \mathcal{M}$, where \mathcal{E}^* is the dual of \mathcal{E} with respect to the Hilbert-Schmidt scalar product.

Note that \mathcal{E}^* is, in particular, a quantum channel, and, thus, the relative entropy in the previous definition is a relative entropy of density matrices and, subsequently, non-negative for every ρ . Moreover, because of its definition, it is clear that σ is a fixed point of \mathcal{E}^* .

The notion introduced above has the advantage with respect to usual conditional relative entropies or conditional relative entropies by expectations that they can be defined for general finite-dimensional von Neumann algebras, and not only for subsystems of certain multipartite systems. However, their main disadvantage is the lack of a meaning of recoverability of states.

These conditional relative entropies will be of relevance in Chapter 8, where they will be used to prove a result of strong quasi-factorization of the relative entropy, i.e., an upper bound

for a conditional relative entropy in terms of two other conditional relative entropies and a multiplicative error term. From the definitions introduced in this chapter for conditional relative entropies, this is the only one that allows, so far, for a result of this nature.

Let us conclude this section with a result of great interest concerning the definition introduced above.

Lemma 6.6.2 — (BCR19b).

Let \mathcal{M} be a matrix algebra, $\mathcal{N} \subset \mathcal{M}$ a matrix subalgebra and $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ a conditional expectation. Then, for any density matrices $\rho, \sigma \in \mathcal{M}$ such that $\mathcal{E}^*(\sigma) = \sigma$, the following holds:

$$D(\rho \parallel \sigma) = D(\rho \parallel \mathcal{E}^*(\rho)) + D(\mathcal{E}^*(\rho) \parallel \sigma). \quad (6.13)$$

Proof. First, denote by $\mathbb{1}$ the identity matrix in \mathcal{M} , and by τ the identity matrix normalized, i.e., divided by its trace. Now, define the state $\sigma_{\text{tr}} := \mathcal{E}^*(\tau)$, so that $\Gamma_{\sigma_{\text{tr}}}^{1/2} \circ \mathcal{E} = \mathcal{E}^* \circ \Gamma_{\sigma_{\text{tr}}}^{1/2}$ (see Proposition 4.3.2). Let us further write $\omega = \mathcal{E}^*(\rho)$ and define $X := \Gamma_{\sigma_{\text{tr}}}^{-1/2}(\sigma)$ and $Y := \Gamma_{\sigma_{\text{tr}}}^{-1/2}(\omega)$. Since σ and ω belong to the fix point set of \mathcal{E}^* , X and Y belong to \mathcal{N} . Then,

$$\begin{aligned} D(\rho \parallel \sigma) &= D(\rho \parallel \omega) + \text{tr}[\rho (\log \omega - \log \sigma)] \\ &= D(\rho \parallel \omega) + \text{tr} \left[\rho \left(\log \left(\Gamma_{\sigma_{\text{tr}}}^{1/2}(X) \right) - \log \left(\Gamma_{\sigma_{\text{tr}}}^{1/2}(Y) \right) \right) \right] \\ &= D(\rho \parallel \omega) + \text{tr}[\rho (\log X - \log Y)] \\ &= D(\rho \parallel \omega) + \text{tr}[\mathcal{E}^*(\rho) (\log X - \log Y)] \\ &= D(\rho \parallel \mathcal{E}^*(\rho)) + D(\mathcal{E}^*(\rho) \parallel \sigma). \end{aligned}$$

where in the third line we used that σ_{tr} commutes with $X, Y \in \mathcal{N}$, so that for example $\log \Gamma_{\sigma_{\text{tr}}}^{1/2}(X) = \log X + \log \sigma_{\text{tr}}$, and the fourth line follows from the fact that $\log X - \log Y \in \mathcal{N}$. ■

R Remark 6.6.3

It is important to remark that this lemma presents a simplified proof for a particular case of [OP93, Theorem 1.13].

Note that the first term in the sum of the RHS of Equation (6.13) is a general conditional relative entropy as defined above. For the particular case of a finite lattice Λ , a subset $A \subset \Lambda$ and considering that the algebra of study is the set of bounded linear operators on A , this lemma implies that the conditional entropy production in A does not depend on the invariant state of the generator chosen.

Indeed, given a local Lindbladian \mathcal{L}_Λ (as the ones that will be considered later in this text, namely the ones associated to the heat-bath and Davies dynamics) in the lattice Λ , and $A \subset \Lambda$ with σ_Λ as a fixed point, recalling Definition 4.5.3, it is clear that, for every $\rho_\Lambda \in \mathcal{S}_\Lambda$, the following holds:

$$\begin{aligned} \text{EP}_A(\rho_\Lambda) &= - \left. \frac{d}{dt} \right|_{t=0} D(\text{e}^{t\mathcal{L}_\Lambda^*}(\rho_\Lambda) \parallel \sigma_\Lambda) \\ &= - \left. \frac{d}{dt} \right|_{t=0} D(\text{e}^{t\mathcal{L}_\Lambda^*}(\rho_\Lambda) \parallel \mathcal{E}^*(\rho_\Lambda)), \end{aligned}$$

since the second term in the RHS of Equation (6.13) does not evolve with time.



7. QUASI-FACTORIZATION OF THE RE

A result of *quasi-factorization* of the relative entropy is an upper bound for the relative entropy of two density matrices in terms of the sum of some conditional relative entropies in certain subsystems, according to the definitions of the previous chapter, and a multiplicative error term which depends only on certain properties of the second state. The motivation for such results, as we saw in Chapter 3, comes from classical spin systems, where a result of quasi-factorization of the classical entropy of a function, proven in both [Ces01] and [DPP02], was essential for the simplification of a seminal result of [MO94b] connecting the mixing time of some Glauber dynamics with the decay of correlations in their Gibbs states, via a positive log-Sobolev constant.

In this chapter, we present several quasi-factorization results for the relative entropy in terms of conditional relative entropies. Depending on two factors, namely the number of subsystems where we condition and their overlap, we will introduce two classes of quasi-factorization results and classify ours into them.

The first class of results concerns bounds for the relative entropy in terms of the sum of two conditional relative entropies in overlapping regions and a multiplicative error term. Namely, for a tripartite Hilbert space $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, we focus on subsystems AB and BC (see Figure 7.1) and prove results of the kind

$$(1 - \xi(\sigma_{ABC}))D(\rho_{ABC}||\sigma_{ABC}) \leq D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}), \quad (\text{QF-Ov})$$

for every $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$, where $\xi(\sigma_{ABC})$ depends only on σ_{ABC} and measures how far σ_{AC} is from $\sigma_A \otimes \sigma_C$.

Results of this class constitute quantum analogues to Lemma 3.5.2, and thus we will mainly focus on them. We will show in the next sections some examples for them, which will be of use in subsequent chapters to obtain examples of positive log-Sobolev constants.

For the second class of results of quasi-factorization, we assume that the systems where we are conditioning the relative entropy in the RHS do not overlap. Thus, imposing strong conditions on the second state, we are able to obtain quasi-factorization results conditioning the

This picture corresponds to an amazing landscape of Florianópolis during the workshop *Q-Turn: Changing paradigms in quantum science*, in November 2018.

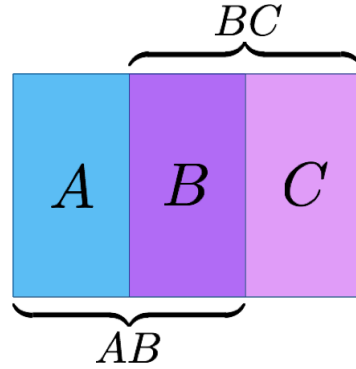


Figure 7.1: Choice of indices in a tripartite Hilbert space $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

relative entropy to a bigger number of regions. More specifically, for a n -partite Hilbert space $\mathcal{H}_{A_1 \dots A_n} = \bigotimes_{i=1}^n \mathcal{H}_{A_i}$, we prove results of the kind

$$(1 - \xi(\sigma_{A_1 \dots A_n}))D(\rho_{A_1 \dots A_n} || \sigma_{A_1 \dots A_n}) \leq \sum_{i=1}^n D_{A_i}(\rho_{A_1 \dots A_n} || \sigma_{A_1 \dots A_n}), \quad (\text{QF-NonOv})$$

for every $\rho_{A_1 \dots A_n}, \sigma_{A_1 \dots A_n} \in \mathcal{S}_{A_1 \dots A_n}$, where $\xi(\sigma_{A_1 \dots A_n})$ depends only on $\sigma_{A_1 \dots A_n}$ and measures in some way how far it is from being a tensor product.

Some examples of results of this class will be presented in the next sections. Indeed, the main result of Section 7.2 will be used in Chapter 9 in the context of quantum spin lattices, as the key step to prove the first non-trivial example of positivity of a log-Sobolev constant in this thesis.

R Remark 7.0.1

It is clear that, whenever one has a result of the first class (QF-Ov), one can construct another one of the second class (QF-NonOv), by conditioning in the RHS only in two regions, just by assuming that $\dim(\mathcal{H}_B)=1$ in the first result.

R Remark 7.0.2

In the next two sections, we will assume that σ is always a tensor product, and, thus, as we have seen in Section 6.4, both definitions of conditional relative entropy and conditional relative entropy by expectations coincide. Hence, except for Section 7.4, all the results of quasi-factorization presented in this chapter concern conditional relative entropies, according to Definition 6.1.1. However, we will present in the next chapter some stronger results of quasi-factorization based on the definition presented in Section 6.6.

Let us conclude the introduction to this chapter remarking that all the results presented here constitute examples of the so-called *weak quasi-factorization of the relative entropy* (see Figure 7.2), since the term appearing in the LHS of the inequality is a relative entropy instead of a conditional relative entropy. However, we will show in the next chapter that some of them can be extended to results of *strong quasi-factorization of the relative entropy* (with a conditional relative entropy appearing in the LHS of the inequality).

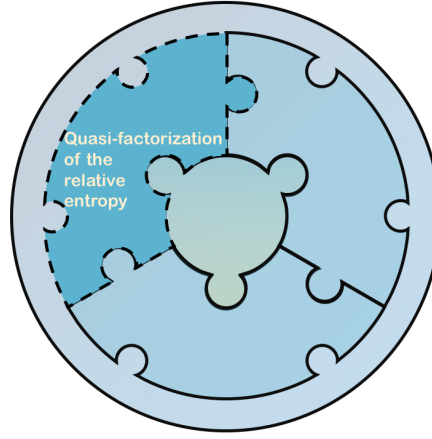


Figure 7.2: Piece associated to the (weak) quasi-factorization of the relative entropy.

7.1 FIRST RESULTS ON QUASI-FACTORIZATION

We will now present some results of quasi-factorization and classify them into the two classes mentioned above. Let us separate the study in different cases in increasing order of difficulty. First, we start showing some results that can be proven directly from the properties in the axiomatic definition of conditional relative entropy. Consider $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. When both states are products, we have the following possibilities:

1. $\dim(\mathcal{H}_B) = 1$, $\rho_{AC} = \rho_A \otimes \rho_C$ and $\sigma_{AC} = \sigma_A \otimes \sigma_C$:

From the property of additivity of Proposition 6.1.4, we can see that

$$D_{A,C}^+(\rho_A \otimes \rho_C || \sigma_A \otimes \sigma_C) = D(\rho_A || \sigma_A) + D(\rho_C || \sigma_C) = D(\rho_A \otimes \rho_C || \sigma_A \otimes \sigma_C).$$

Hence, in this case,

$$D(\rho_{AC} || \sigma_{AC}) = D_A(\rho_{AC} || \sigma_{AC}) + D_C(\rho_{AC} || \sigma_{AC})$$

constitutes the simplest result of quasi-factorization of both (QF-Ov) and (QF-NonOv).

2. Arbitrary dimension of \mathcal{H}_B , $\rho_{ABC} = \rho_A \otimes \rho_B \otimes \rho_C$ and $\sigma_{ABC} = \sigma_A \otimes \sigma_B \otimes \sigma_C$:

This case is an extension of the previous one. We have

$$D(\rho_{ABC} || \sigma_{ABC}) = D_A(\rho_{ABC} || \sigma_{ABC}) + D_B(\rho_{ABC} || \sigma_{ABC}) + D_C(\rho_{ABC} || \sigma_{ABC}),$$

which is clearly a result of (QF-NonOv).

3. In general, for $n \in \mathbb{N}$, $\mathcal{H}_{A_1 \dots A_n} = \bigotimes_{i=1}^n \mathcal{H}_{A_i}$, $\rho_{A_1 \dots A_n} = \bigotimes_{i=1}^n \rho_{A_i}$ and $\sigma_{A_1 \dots A_n} = \bigotimes_{i=1}^n \sigma_{A_i}$:

This case is a generalization of the previous one. Because of the property of semi-additivity, we clearly have

$$D(\rho_{A_1 \dots A_n} || \sigma_{A_1 \dots A_n}) = \sum_{i=1}^n D_{A_i}(\rho_{A_1 \dots A_n} || \sigma_{A_1 \dots A_n}),$$

which is a result of (QF-NonOv).

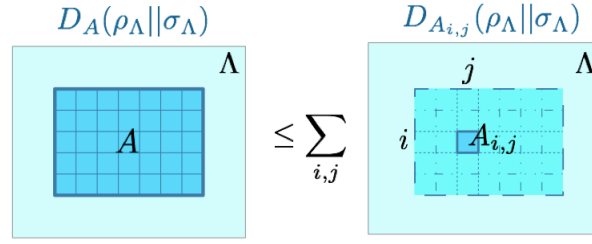


Figure 7.3: Graphical representation for the result of quasi-factorization obtained under the assumption of both ρ and σ tensor products.

4. The regions AB and BC , $\rho_{ABC} = \rho_A \otimes \rho_B \otimes \rho_C$ and $\sigma_{ABC} = \sigma_A \otimes \sigma_B \otimes \sigma_C$:

Under these assumptions we have

$$D_{AB,BC}^+(\rho_{ABC} || \sigma_{ABC}) = D(\rho_A || \sigma_A) + 2D(\rho_B || \sigma_B) + D(\rho_C || \sigma_C) \geq D(\rho_{ABC} || \sigma_{ABC}),$$

where the last inequality comes from the additivity and non-negativity of the relative entropy. Hence,

$$D(\rho_{ABC} || \sigma_{ABC}) \leq D_{AB}(\rho_{ABC} || \sigma_{ABC}) + D_{BC}(\rho_{ABC} || \sigma_{ABC})$$

constitutes a result of (QF-Ov).

R Remark 7.1.1

Note that, in the previous four cases, the error term $\xi(\sigma_{ABC})$ does not appear in the quasi-factorization result. This is something reasonable, since this term should measure how far σ_{AC} is from $\sigma_A \otimes \sigma_C$ and, by assumption, in this case, this “distance” is zero.

The four cases addressed above can be represented, in general, by the graphical expression of the quasi-factorization that appears in Figure 7.3.

Let us now consider again a tripartite Hilbert space, relax the assumption on ρ_{ABC} and assume only $\sigma_{ABC} = \sigma_A \otimes \sigma_B \otimes \sigma_C$. Without imposing any condition on ρ_{ABC} , we are not able to obtain results of quasi-factorization from properties (1)-(3) in Definition 6.1.1 (and, thus, fulfilled by both the conditional relative entropy and the conditional relative entropy by expectations) as we have just done above.

However, for the conditional relative entropy (and the conditional relative entropy by expectations), we have the following property: If $\sigma_{ABC} = \sigma_A \otimes \sigma_B \otimes \sigma_C$, we saw in Section 6.4 that the following holds:

$$D_A(\rho_{ABC} || \sigma_{ABC}) = I_\rho(A : BC) + D(\rho_A || \sigma_A). \quad (7.1)$$

Taking this property into account, for subsystems AB and BC we present another quasi-factorization result of the kind (QF-Ov) (see Figure 7.4).

Proposition 7.1.2 Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$ such that $\sigma_{ABC} = \sigma_A \otimes \sigma_B \otimes \sigma_C$. The following inequality holds:

$$D(\rho_{ABC} || \sigma_{ABC}) \leq D_{AB}(\rho_{ABC} || \sigma_{ABC}) + D_{BC}(\rho_{ABC} || \sigma_{ABC}). \quad (7.2)$$

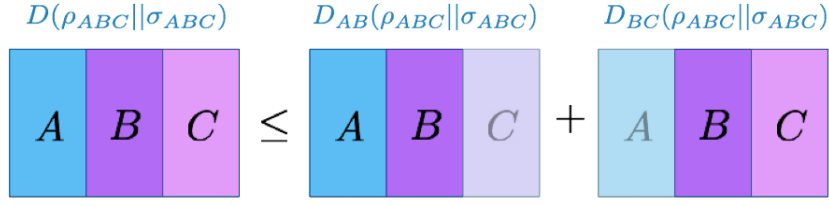


Figure 7.4: Graphical representation for the result of quasi-factorization of the kind (QF-Ov) obtained under the assumption of σ_{ABC} tensor product.

Proof. Due to property (7.1), for both definitions we have:

$$\begin{aligned}
 D_{AB,BC}^+(\rho_{ABC} || \sigma_A \otimes \sigma_B \otimes \sigma_C) &= D_{AB}(\rho_{ABC} || \sigma_A \otimes \sigma_B \otimes \sigma_C) + D_{BC}(\rho_{ABC} || \sigma_A \otimes \sigma_B \otimes \sigma_C) \\
 &= I_\rho(AB : C) + D(\rho_{AB} || \sigma_A \otimes \sigma_B) + I_\rho(BC : A) + D(\rho_{BC} || \sigma_B \otimes \sigma_C).
 \end{aligned}$$

Now, because of monotonicity of the relative entropy with respect to the partial trace and additivity,

$$\begin{aligned}
 D(\rho_{AB} || \sigma_A \otimes \sigma_B) + D(\rho_{BC} || \sigma_B \otimes \sigma_C) &\geq D(\rho_A || \sigma_A) + D(\rho_{BC} || \sigma_B \otimes \sigma_C) \\
 &\geq D(\rho_A \otimes \rho_{BC} || \sigma_A \otimes \sigma_B \otimes \sigma_C),
 \end{aligned}$$

and adding this term to $I_\rho(BC : A)$, we have:

$$\begin{aligned}
 I_\rho(BC : A) + D(\rho_A \otimes \rho_{BC} || \sigma_A \otimes \sigma_B \otimes \sigma_C) &= D(\rho_{ABC} || \rho_A \otimes \rho_{BC}) + D(\rho_A \otimes \rho_{BC} || \sigma_A \otimes \sigma_B \otimes \sigma_C) \\
 &= D(\rho_{ABC} || \sigma_A \otimes \sigma_B \otimes \sigma_C).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 D_{AB,BC}^+(\rho_{ABC} || \sigma_A \otimes \sigma_B \otimes \sigma_C) &\geq I_\rho(AB : C) + I_\rho(BC : A) + D(\rho_A \otimes \rho_{BC} || \sigma_A \otimes \sigma_B \otimes \sigma_C) \\
 &= I_\rho(AB : C) + D(\rho_{ABC} || \sigma_A \otimes \sigma_B \otimes \sigma_C) \\
 &\geq D(\rho_{ABC} || \sigma_A \otimes \sigma_B \otimes \sigma_C).
 \end{aligned}$$

■

We will show a more general version of this proposition, when σ is not a tensor product, in Section 7.3.

Considering now the regions A , B and C , and again due to property (7.1), we can prove the following result of (QF-NonOv) (see Figure 7.5).

Proposition 7.1.3 Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$ such that $\sigma_{ABC} = \sigma_A \otimes \sigma_B \otimes \sigma_C$. The following inequality holds:

$$D(\rho_{ABC} || \sigma_{ABC}) \leq D_A(\rho_{ABC} || \sigma_{ABC}) + D_B(\rho_{ABC} || \sigma_{ABC}) + D_C(\rho_{ABC} || \sigma_{ABC}). \quad (7.3)$$

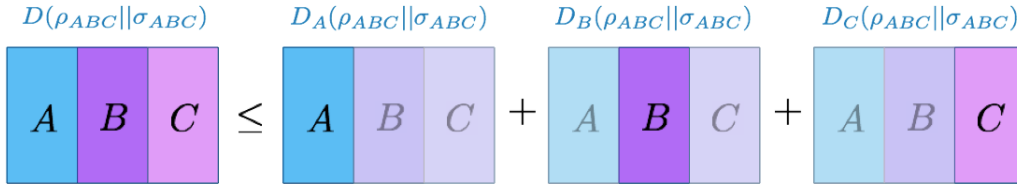


Figure 7.5: Graphical representation for the result of quasi-factorization of the kind (QF-NonOv) obtained under the assumption of σ_{ABC} tensor product.

Proof. Analogously to the definition of $D_{A,B}^+$, we define $D_{A,B,C}^+$:

$$\begin{aligned} D_{A,B,C}^+(\rho_{ABC} || \sigma_A \otimes \sigma_B \otimes \sigma_C) &:= \\ &= D_A(\rho_{ABC} || \sigma_A \otimes \sigma_B \otimes \sigma_C) + D_B(\rho_{ABC} || \sigma_A \otimes \sigma_B \otimes \sigma_C) + D_C(\rho_{ABC} || \sigma_A \otimes \sigma_B \otimes \sigma_C) \\ &= I_\rho(A : BC) + D(\rho_A || \sigma_A) + I_\rho(B : AC) + D(\rho_B || \sigma_B) + I_\rho(C : AB) + D(\rho_C || \sigma_C). \end{aligned}$$

Now, we have the following lower bound for the mutual informations:

$$\begin{aligned} I_\rho(A : BC) + I_\rho(B : AC) + I_\rho(C : AB) &= \\ &= \text{tr}[\rho_{ABC} (3 \log \rho_{ABC} - \log \rho_A \otimes \rho_{BC} - \log \rho_B \otimes \rho_{AC} - \log \rho_C \otimes \rho_{AB})] \\ &= \text{tr}[\rho_{ABC} (3 \log \rho_{ABC} - \log \rho_A \otimes \rho_B \otimes \rho_C - \log \rho_{BC} - \log \rho_{AC} - \log \rho_{AB})] \\ &= D(\rho_{ABC} || \rho_A \otimes \rho_B \otimes \rho_C) + \text{tr}[\rho_{ABC} (2 \log \rho_{ABC} - \log \rho_{BC} - \log \rho_{AC} - \log \rho_{AB})] \\ &= D(\rho_{ABC} || \rho_A \otimes \rho_B \otimes \rho_C) + \text{tr}[\rho_{ABC} (\log \rho_{ABC} - \log \rho_{BC} - \log \rho_{AC} + \log \rho_C)] \\ &\quad + \text{tr}[\rho_{ABC} (\log \rho_{ABC} - \log \rho_{AB} \otimes \rho_C)] \\ &\geq D(\rho_{ABC} || \rho_A \otimes \rho_B \otimes \rho_C) + I_\rho(AB : C) \\ &\geq D(\rho_{ABC} || \rho_A \otimes \rho_B \otimes \rho_C), \end{aligned}$$

where we have used strong subadditivity for the von Neumann entropy and non-negativity for the relative entropy.

Therefore,

$$\begin{aligned} D_{A,B,C}^+(\rho_{ABC} || \sigma_A \otimes \sigma_B \otimes \sigma_C) &\geq D(\rho_{ABC} || \rho_A \otimes \rho_B \otimes \rho_C) + D(\rho_A || \sigma_A) + D(\rho_B || \sigma_B) + D(\rho_C || \sigma_C) \\ &= D(\rho_{ABC} || \sigma_A \otimes \sigma_B \otimes \sigma_C). \end{aligned}$$

■

By considering two non-overlapping subregions instead of three in the RHS, the quasi-factorization result of Section 7.4 constitutes a generalization of this proposition when σ is not a tensor product, for the conditional relative entropy by expectations. Moreover, if in the quasi-factorization result of Section 7.3 we assume $\dim(\mathcal{H}_B) = 1$, that result also constitutes a generalization of this proposition for two subregions when σ is not necessarily a tensor product, for the conditional relative entropy. In both results, we will need the explicit expressions of conditional relative entropy and conditional relative entropy by expectations, respectively, oppositely to the cases mentioned above, where we obtained quasi-factorization results just from some properties of the definitions.

Concerning the number of subregions (and, thus, number of conditional relative entropies in the RHS), this proposition can be generalized to n -partite Hilbert spaces. We will show that in the following section.

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{i,j} i D_{A_{i,j}}(\rho_\Lambda || \sigma_\Lambda)$$

Figure 7.6: Graphical representation for the most general result of (weak) quasi-factorization under the assumption of σ_Λ tensor product.

7.2 QUASI-FACTORIZATION FOR σ A TENSOR PRODUCT

In this section, we show that, imposing strong conditions on the second state, we manage to prove a quasi-factorization of the relative entropy in terms of many more conditional relative entropies (see Figure 7.6). Instead of tripartite states, we consider now multipartite ones. To simplify notation, let $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$ be a multipartite Hilbert space, and let $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$. We will prove that the relative entropy of both states is upper bounded by the sum of all the conditional relative entropies in every $x \in \Lambda$. The multiplicative error term again disappears, since the state considered here is a tensor product.

Although we state it here as an upper bound for a relative entropy of two states, this result constitutes an example of strong quasi-factorization, as we will see in the next chapter (see Section 8.1).

Theorem 7.2.1 — QUASI-FACTORIZATION FOR σ A TENSOR PRODUCT, (CLP18a).

Let \mathcal{H}_Λ be a multipartite Hilbert space and let $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$ such that $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$. The following inequality holds:

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda). \quad (7.4)$$

The proof of this theorem is based on the following result:

Lemma 7.2.2 Let Λ be a finite set, \mathcal{H}_Λ a multipartite Hilbert space and $\rho_\Lambda \in \mathcal{S}_\Lambda$. The following inequality holds:

$$S(\rho_\Lambda) \geq \sum_{x \in \Lambda} S(x|x^c)_\rho, \quad (7.5)$$

where $S(x|x^c)_\rho$ is the conditional entropy:

$$S(x|x^c)_\rho = S(\rho_\Lambda) - S(\rho_{x^c}).$$

This result constitutes a particular case of the quantum version of Shearer's inequality. It has been proven in several papers, such as [MFW16b] and [JP16], where the proof is based in the *strong subadditivity property* of the von Neumann entropy [LR73].

We can now proceed to the proof of Theorem 7.2.1.

Proof. Let us rewrite Equation (7.4) as:

$$D(\rho_\Lambda || \sigma_\Lambda) - \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda) \leq 0, \quad (7.6)$$

where $D_x(\rho_\Lambda || \sigma_\Lambda)$ is given by

$$D_x(\rho_\Lambda || \sigma_\Lambda) = D(\rho_\Lambda || \sigma_\Lambda) - D(\rho_{x^c} || \sigma_{x^c}).$$

Hence, the left-hand side of the previous inequality can be expressed by:

$$\begin{aligned} D(\rho_\Lambda || \sigma_\Lambda) - \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda) &= \\ &= (1 - |\Lambda|) D(\rho_\Lambda || \sigma_\Lambda) + \sum_{x \in \Lambda} D(\rho_{x^c} || \sigma_{x^c}) \\ &= (1 - |\Lambda|) \text{tr} \left[\rho_\Lambda \left(\log \rho_\Lambda - \log \sigma_\Lambda + \frac{1}{1 - |\Lambda|} \sum_{x \in \Lambda} \log \rho_{x^c} - \frac{1}{1 - |\Lambda|} \sum_{x \in \Lambda} \log \sigma_{x^c} \right) \right]. \end{aligned}$$

If we now consider only the terms concerning σ_Λ , using the fact that $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$ we have:

$$\begin{aligned} (|\Lambda| - 1) \log \sigma_\Lambda - \sum_{x \in \Lambda} \log \sigma_{x^c} &= \\ &= (|\Lambda| - 1) \sum_{x \in \Lambda} \log \sigma_x - \sum_{x \in \Lambda} \sum_{y \neq x} \log \sigma_y \\ &= (|\Lambda| - 1) \sum_{x \in \Lambda} \log \sigma_x - (|\Lambda| - 1) \sum_{x \in \Lambda} \log \sigma_x = 0. \end{aligned}$$

Therefore,

$$D(\rho_\Lambda || \sigma_\Lambda) - \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda) = (1 - |\Lambda|) \text{tr} \left[\rho_\Lambda \left(\log \rho_\Lambda + \frac{1}{1 - |\Lambda|} \sum_{x \in \Lambda} \log \rho_{x^c} \right) \right],$$

and, thus, Equation (7.6) can be rewritten as

$$(|\Lambda| - 1) S(\rho_\Lambda) - \sum_{x \in \Lambda} S(\rho_{x^c}) \leq 0, \quad (7.7)$$

where we are denoting by $S(\rho_\Lambda)$ the von Neumann entropy of ρ_Λ .

Finally, recalling that the conditional entropy is defined as

$$S(x|x^c)_\rho = S(\rho_\Lambda) - S(\rho_{x^c}),$$

expression (7.7) is equivalent to (7.5), finishing thus the proof. \blacksquare

R Remark 7.2.3

Note that, as we mentioned above, since we are assuming for this result that σ_Λ is a tensor product, this quasi-factorization holds equally for the conditional relative entropy and the conditional relative entropy by expectations, respectively.

In Chapter 9, we will show how this result of quasi-factorization of the kind (QF-NonOv) can be used to obtain a log-Sobolev constant for the heat-bath dynamics, when the fixed state of the evolution is product.

When σ_{ABC} is not a product state, the situation is a bit more complicated. Now, a term $\xi(\sigma_{ABC})$ should appear, measuring how far σ_{AC} is from a product state, as a multiplicative error term. In the following two sections, we provide two results of quasi-factorization of the relative entropy, one for the conditional relative entropy and another (weaker) one for the conditional relative entropy by expectations. As we have mentioned above, for both results we will need the explicit expressions for conditional relative entropy and conditional relative entropy by expectations, respectively, as we will not be able to obtain them from the properties in the definitions.

Figure 7.7: Graphical representation for a result of quasi-factorization for the conditional relative entropy for arbitrary ρ_{ABC} and σ_{ABC} .

7.3 QUASI-FACTORIZATION FOR THE CONDITIONAL RELATIVE ENTROPY

In this section, we present a quasi-factorization result for the relative entropy in terms of conditional relative entropies. We need to consider some overlap in the regions where we are conditioning the relative entropies of the RHS due to the envisaged applications in quantum many body systems (see Figure 7.7). In virtue of the identification between quantum and classical spin systems mentioned in Section 6.5, this result can be seen as the quantum analogue of Lemma 3.5.2. We will show that this result is equivalent to [CLP18b, Theorem 1], which appears in this thesis as Theorem 5.0.1.

Theorem 7.3.1 — QUASI-FACTORIZATION FOR THE CRE, (CLP18a).

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ be a tripartite Hilbert space and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. Then, the following inequality holds

$$(1 - 2\|H(\sigma_{AC})\|_\infty)D(\rho_{ABC}||\sigma_{ABC}) \leq D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}), \quad (7.8)$$

where

$$H(\sigma_{AC}) = \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \sigma_{AC} \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC}.$$

Note that $H(\sigma_{AC}) = 0$ if σ_{AC} is a tensor product between A and C .

Proof. It is enough to prove the equivalence between Theorem 5.0.1 and Theorem 7.3.1.

Th. 7.3.1 \Rightarrow Th. 5.0.1: Let $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. Then,

$$\begin{aligned} (1 - 2\|H(\sigma_{AC})\|_\infty)D(\rho_{ABC}||\sigma_{ABC}) &\leq D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \\ &= 2D(\rho_{ABC}||\sigma_{ABC}) - D(\rho_C||\sigma_C) - D(\rho_A||\sigma_A). \end{aligned}$$

Rewriting this to have something more similar to inequality (5), we have

$$(1 + 2\|H(\sigma_{AC})\|_\infty)D(\rho_{ABC}||\sigma_{ABC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C),$$

so considering a particular case in which the dimension of \mathcal{H}_B is 1 (thus, $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_C$), we have inequality (5).

Th. 5.0.1 \Rightarrow Th. 7.3.1: From the monotonicity of the relative entropy, we know that

$$D(\rho_{ABC}||\sigma_{ABC}) \geq D(\rho_{AC}||\sigma_{AC}),$$

and using this together with inequality (5), we have

$$(1 + 2\|H(\sigma_{AC})\|_\infty)D(\rho_{ABC}||\sigma_{ABC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C),$$

which we have just seen that is a reformulation of inequality (7.8). ■

R Remark 7.3.2

It is clear that Proposition 7.1.2 constitutes a particular case of this theorem where the multiplicative error term disappears, since in that case we were considering σ_{ABC} a tensor product.

Note that, as opposed to the situation in the previous section, this result cannot be extended easily to a strong quasi-factorization of the relative entropy. Indeed, it constitutes a “weaker” result than those in which the upper bound is provided for a conditional relative entropy, and we represent it by the image that appears in Figure 7.2.

Analogously to what we mentioned in the previous section concerning the result of quasi-factorization for σ a tensor product, this result of quasi-factorization of the kind (QF-Ov) will be further used in Chapter 10 to obtain positivity for certain logarithmic Sobolev constants, under a sufficiently strong assumption on the decay of correlations in σ .

7.4 QUASI-FACTORIZATION FOR THE CRE BY EXPECTATIONS

In this section, we consider conditional relative entropies for expectations instead of usual conditional relative entropies. For them, we can prove the following result, which is an example of (QF-NonOv) for two subregions (see Figure 7.8). It also constitutes an example of the case that appears in Figure 7.2.

Theorem 7.4.1 — QUASI-FACTORIZATION FOR THE CRE BY EXPECTATIONS, (CLP18a).

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite Hilbert space and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. Then, the following inequality holds

$$(1 - \xi(\sigma_{AB}))D(\rho_{AB}||\sigma_{AB}) \leq D_A^E(\rho_{AB}||\sigma_{AB}) + D_B^E(\rho_{AB}||\sigma_{AB}), \quad (7.9)$$

where

$$\xi(\sigma_{AB}) = 2(E_1(t) + E_2(t)),$$

and

$$E_1(t) = \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{-\frac{1+i}{2}} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-\frac{1+i}{2}} - \mathbb{1}_{AB} \right\|_{\infty} \left\| \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \sigma_B^{-1/2} \right\|_{\infty},$$

$$E_2(t) = \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{-\frac{1-i}{2}} \sigma_{AB}^{\frac{1+i}{2}} \sigma_A^{-\frac{1-i}{2}} - \mathbb{1}_{AB} \right\|_{\infty},$$

with

$$\beta_0(t) = \frac{\pi}{2} (\cosh(\pi t) + 1)^{-1}.$$

Note that $\xi(\sigma_{AB}) = 0$ if σ_{AB} is a tensor product between A and B .

This proof can be split into four steps. The first part of the proof is analogous to the one of Theorem 5.0.1, but we include it here for the sake of clearness. However, from the second half of the second step, the proof gets much more complicated, leading to the error term shown in the statement of the theorem, which, despite going in the same spirit than its analogue in Theorem 7.3.1, is less intuitive.

The figure shows a graphical representation of the inequality $D(\rho_{AB}||\sigma_{AB}) \leq \xi \left(\begin{smallmatrix} \sigma_{AB} \\ \begin{smallmatrix} A & B \end{smallmatrix} \end{smallmatrix} \leftrightarrow \begin{smallmatrix} \sigma_A \otimes \sigma_B \\ \begin{smallmatrix} A & B \end{smallmatrix} \end{smallmatrix} \right) \left(\begin{smallmatrix} D_A^E(\rho_{AB}||\sigma_{AB}) \\ \begin{smallmatrix} A & B \end{smallmatrix} \end{smallmatrix} + \begin{smallmatrix} D_B^E(\rho_{AB}||\sigma_{AB}) \\ \begin{smallmatrix} A & B \end{smallmatrix} \end{smallmatrix} \right)$. On the left, a blue box labeled 'A' and a purple box labeled 'B' are side-by-side, with the label $D(\rho_{AB}||\sigma_{AB})$ above them. In the middle, a double-headed arrow connects two boxes. The first box has a small box labeled σ_{AB} above it and contains a blue box 'A' and a purple box 'B'. The second box has a small box labeled $\sigma_A \otimes \sigma_B$ above it and also contains a blue box 'A' and a purple box 'B'. On the right, there are two boxes. The first box has a label $D_A^E(\rho_{AB}||\sigma_{AB})$ above it and contains a blue box 'A' and a purple box 'B'. The second box has a label $D_B^E(\rho_{AB}||\sigma_{AB})$ above it and contains a light blue box 'A' and a purple box 'B'. A plus sign is between these two boxes, and the entire group is enclosed in large parentheses.

Figure 7.8: Graphical representation for a result of quasi-factorization for the conditional relative entropy by expectations for arbitrary ρ_{AB} and σ_{AB} .

7.4.1 STEP 1: ADDITIVE ERROR TERM FOR THE DIFFERENCE OF RELATIVE ENTROPIES

Analogously to Step 5.1.1, we can prove now:

Step 7.4.2 For density matrices $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$, it holds that

$$D(\rho_{AB}||\sigma_{AB}) \leq D_A^E(\rho_{AB}||\sigma_{AB}) + D_B^E(\rho_{AB}||\sigma_{AB}) + \log \text{tr} M, \quad (7.10)$$

where $M = \exp[-\log \sigma_{AB} + \log \mathbb{E}_A^*(\rho_{AB}) + \log \mathbb{E}_B^*(\rho_{AB})]$ and equality holds (both sides being equal to zero) if $\rho_{AB} = \sigma_{AB}$.

Moreover, if $\sigma_{AB} = \sigma_A \otimes \sigma_B$, then $\log \text{tr} M = 0$.

From the definition of conditional relative entropy by expectations it follows that:

$$\begin{aligned} D(\rho_{AB}||\sigma_{AB}) - D_A^E(\rho_{AB}||\sigma_{AB}) - D_B^E(\rho_{AB}||\sigma_{AB}) &= \\ &= D(\rho_{AB}||\sigma_{AB}) - D(\rho_{AB}||\mathbb{E}_A^*(\rho_{AB})) - D(\rho_{AB}||\mathbb{E}_B^*(\rho_{AB})) \\ &= \text{tr} \left[\rho_{AB} \left(-\log \rho_{AB} - \underbrace{-\log \sigma_{AB} + \log \mathbb{E}_A^*(\rho_{AB}) + \log \mathbb{E}_B^*(\rho_{AB})}_{\log M} \right) \right] \\ &= -D(\rho_{AB}||M). \end{aligned}$$

Now, since $\text{tr}[M] \neq 1$ in general,

$$D(\rho_{AB}||M) = D(\rho_{AB}||M/\text{tr}[M]) - \log \text{tr}[M] \geq -\log \text{tr}[M],$$

due to the non-negativity property of the relative entropy.

If $\rho_{AB} = \sigma_{AB}$, $\mathbb{E}_A^*(\rho_{AB}) = \sigma_{AB}$, and the same for \mathbb{E}_B^* , so $\log M = \log \sigma_{AB}$ and both sides are equal to zero. Also, if $\sigma_{AB} = \sigma_A \otimes \sigma_B$, we have $\mathbb{E}_A^*(\rho_{AB}) = \sigma_A \otimes \rho_B$ and $\mathbb{E}_B^*(\rho_{AB}) = \rho_A \otimes \sigma_B$, so $M = \rho_A \otimes \rho_B$. Hence, $\log \text{tr} M = 0$.

7.4.2 STEP 2: ERROR TERM WITH LIEB'S EXTENSION OF GOLDEN-THOMPSON

In this and the next steps, we focus on bounding $\log \text{tr} M$ in terms of the relative entropy between ρ_{AB} and σ_{AB} multiplied by a term that only depends on how far σ_{AB} is from a tensor product. First, we will bound this term by the a term in the same spirit than the analogue in Step 5.2.5. We will make use again of Theorem 5.2.1 and Lemma 5.2.3, concerning Lieb's extension of Golden-Thompson inequality and Sutter, Berta and Tomamichel's rotated expression for Lieb's pseudo-inversion operator using multivariate trace inequalities, respectively.

Step 7.4.3 With the same notation of Step 7.4.2, we have that

$$\log \operatorname{tr} M \leq \int_{-\infty}^{+\infty} dt \beta_0(t) \operatorname{tr} \left[\sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_B^{-1/2} (\rho_B - \sigma_B) \sigma_B^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right], \quad (7.11)$$

with

$$\beta_0(t) = \frac{\pi}{2} (\cosh(\pi t) + 1)^{-1}.$$

Proof. Applying Theorem 5.2.1 to inequality (7.10), we have

$$\begin{aligned} \operatorname{tr} M &= \operatorname{tr} \left[\exp \left(\underbrace{-\log \sigma_{AB}}_f + \underbrace{\log \mathbb{E}_A^*(\rho_{AB})}_h + \underbrace{\log \mathbb{E}_B^*(\rho_{AB})}_g \right) \right] \\ &\leq \operatorname{tr} [\mathbb{E}_A^*(\rho_{AB}) \mathcal{T}_{\sigma_{AB}}(\mathbb{E}_B^*(\rho_{AB}))], \end{aligned}$$

and by virtue of Lemma 5.2.3,

$$\operatorname{tr} M \leq \operatorname{tr} \left[\mathbb{E}_A^*(\rho_{AB}) \int_{-\infty}^{+\infty} dt \beta_0(t) \sigma_{AB}^{\frac{-1-i}{2}} \mathbb{E}_B^*(\rho_{AB}) \sigma_{AB}^{\frac{-1+i}{2}} \right].$$

Now, replacing the values of $\mathbb{E}_A^*(\rho_{AB})$ and $\mathbb{E}_B^*(\rho_{AB})$, and using the linearity of the trace, we have

$$\begin{aligned} \operatorname{tr} M &\leq \int_{-\infty}^{+\infty} dt \beta_0(t) \operatorname{tr} \left[\mathbb{E}_A^*(\rho_{AB}) \sigma_{AB}^{\frac{-1-i}{2}} \mathbb{E}_B^*(\rho_{AB}) \sigma_{AB}^{\frac{-1+i}{2}} \right] \\ &= \int_{-\infty}^{+\infty} dt \beta_0(t) \operatorname{tr} \left[\sigma_{AB}^{1/2} \sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \sigma_{AB}^{1/2} \sigma_{AB}^{\frac{-1-i}{2}} \sigma_{AB}^{1/2} \sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \sigma_{AB}^{1/2} \sigma_{AB}^{\frac{-1+i}{2}} \right] \\ &= \int_{-\infty}^{+\infty} dt \beta_0(t) \operatorname{tr} \left[\sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] \end{aligned}$$

If we subtract σ_B from ρ_B and σ_A from ρ_A in the term inside the integral of the previous expression, we have

$$\begin{aligned} &\operatorname{tr} \left[\sigma_B^{-1/2} (\rho_B - \sigma_B) \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] = \\ &= \operatorname{tr} \left[\sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] \\ &\quad + \operatorname{tr} \left[\sigma_B^{-1/2} \sigma_B \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} \sigma_A \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] \\ &\quad - \operatorname{tr} \left[\sigma_B^{-1/2} \sigma_B \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] \\ &\quad - \operatorname{tr} \left[\sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} \sigma_A \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right], \end{aligned}$$

where these four terms can be simplified in the following way:

$$\begin{aligned}
& \operatorname{tr} \left[\sigma_B^{-1/2} (\rho_B - \sigma_B) \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] = \\
& = \left(\operatorname{tr} \left[\sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] + \operatorname{tr} \left[\sigma_{AB}^{\frac{1-i}{2}} \sigma_{AB}^{\frac{1+i}{2}} \right] \right) \\
& \quad - \left(\operatorname{tr} \left[\sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] + \operatorname{tr} \left[\sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_{AB}^{\frac{1+i}{2}} \right] \right) \\
& = \left(\operatorname{tr} \left[\sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] + 1 \right) \\
& \quad - \left(\operatorname{tr} \left[\sigma_A \sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \right] + \operatorname{tr} \left[\sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \right] \right) \\
& = \operatorname{tr} \left[\sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] + 1 - 1 - 1,
\end{aligned}$$

using some properties of the trace, such as its linearity, cyclicity and the fact that if $f_A \in \mathcal{A}_A$ and $g_{AB} \in \mathcal{S}_{AB}$ then

$$\operatorname{tr}[f_A g_{AB}] = \operatorname{tr}[f_A g_{AB}].$$

Therefore, we have the following equality:

$$\begin{aligned}
& \operatorname{tr} \left[\sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} \rho_A \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] = \\
& = \operatorname{tr} \left[\sigma_B^{-1/2} (\rho_B - \sigma_B) \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] + 1,
\end{aligned}$$

and hence

$$\operatorname{tr} M \leq \int_{-\infty}^{+\infty} dt \beta_0(t) \left(\operatorname{tr} \left[\sigma_B^{-1/2} (\rho_B - \sigma_B) \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] + 1 \right)$$

If we now use the following inequality for positive real numbers

$$\log(x) \leq x - 1,$$

and the monotonicity of the logarithm and the fact that $\beta_0(t)$ integrates 1, we can then conclude

$$\begin{aligned}
& \log \operatorname{tr} M \leq \\
& \leq \log \left[\int_{-\infty}^{+\infty} dt \beta_0(t) \left(\operatorname{tr} \left[\sigma_B^{-1/2} (\rho_B - \sigma_B) \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] + 1 \right) \right] \\
& = \log \left[\int_{-\infty}^{+\infty} dt \beta_0(t) \operatorname{tr} \left[\sigma_B^{-1/2} (\rho_B - \sigma_B) \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] + 1 \right] \\
& \leq \int_{-\infty}^{+\infty} dt \beta_0(t) \operatorname{tr} \left[\sigma_B^{-1/2} (\rho_B - \sigma_B) \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right].
\end{aligned}$$

■

7.4.3 STEP 3: SPLITTING THE ERROR TERM INTO TWO PARTS

In the third step of the proof, we split the error term into two parts, each one of which will be studied separately in the last step of the proof.

Step 7.4.4 With the same notation of the previous steps,

$$\mathrm{tr} \left[\sigma_B^{-1/2} (\rho_B - \sigma_B) \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] = \xi_1 + \xi_2, \quad (7.12)$$

where

$$\begin{aligned} \xi_1 &= \mathrm{tr} \left[T_B \left(\sigma_{AB}^{\frac{1-i}{2}} - (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} \right) T_A \sigma_{AB}^{\frac{1+i}{2}} \right], \\ \xi_2 &= \mathrm{tr} \left[T_B (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} T_A \left(\sigma_{AB}^{\frac{1+i}{2}} - (\sigma_A \otimes \sigma_B)^{\frac{1+i}{2}} \right) \right], \end{aligned}$$

for certain observables $T_A \in \mathcal{A}_A$ and $T_B \in \mathcal{A}_B$.

Note that both ξ_1 and ξ_2 vanish when σ_{AB} is a tensor product.

Proof. Let us first write

$$T_A := \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2},$$

$$T_B := \sigma_B^{-1/2} (\rho_B - \sigma_B) \sigma_B^{-1/2},$$

to simplify notation. Hence

$$\mathrm{tr} \left[\sigma_B^{-1/2} (\rho_B - \sigma_B) \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \right] = \mathrm{tr} \left[T_B \sigma_{AB}^{\frac{1-i}{2}} T_A \sigma_{AB}^{\frac{1+i}{2}} \right].$$

Now, we add and subtract $(\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}}$ to $\sigma_{AB}^{\frac{1-i}{2}}$ and $(\sigma_A \otimes \sigma_B)^{\frac{1+i}{2}}$ to $\sigma_{AB}^{\frac{1+i}{2}}$, respectively. We will later use some combinations of these terms in the error terms, so that we recover the fact that the error terms vanish whenever σ_{AB} is a tensor product.

$$\begin{aligned} \mathrm{tr} \left[T_B \sigma_{AB}^{\frac{1-i}{2}} T_A \sigma_{AB}^{\frac{1+i}{2}} \right] &= \mathrm{tr} \left[T_B \left(\sigma_{AB}^{\frac{1-i}{2}} - (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} + (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} \right) \right. \\ &\quad \cdot \left. T_A \left(\sigma_{AB}^{\frac{1+i}{2}} - (\sigma_A \otimes \sigma_B)^{\frac{1+i}{2}} + (\sigma_A \otimes \sigma_B)^{\frac{1+i}{2}} \right) \right] \\ &= \underbrace{\mathrm{tr} \left[T_B \left(\sigma_{AB}^{\frac{1-i}{2}} - (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} \right) T_A \sigma_{AB}^{\frac{1+i}{2}} \right]}_{\xi_1} \\ &\quad + \underbrace{\mathrm{tr} \left[T_B (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} T_A \left(\sigma_{AB}^{\frac{1+i}{2}} - (\sigma_A \otimes \sigma_B)^{\frac{1+i}{2}} \right) \right]}_{\xi_2} \\ &\quad + \underbrace{\mathrm{tr} \left[T_B (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} T_A (\sigma_A \otimes \sigma_B)^{\frac{1+i}{2}} \right]}_{\xi_3}. \end{aligned}$$

There is only left to prove that ξ_3 vanishes. For that, let us replace again the values of T_A and

T_B in the expression of ξ_3 .

$$\begin{aligned}
\xi_3 &= \text{tr} \left[\sigma_B^{-1/2} (\rho_B - \sigma_B) \sigma_B^{-1/2} (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} (\sigma_A \otimes \sigma_B)^{\frac{1+i}{2}} \right] \\
&= \text{tr} \left[\sigma_B^{\frac{1+i}{2}} \sigma_B^{-1/2} (\rho_B - \sigma_B) \sigma_B^{-1/2} \sigma_B^{\frac{1-i}{2}} \sigma_A^{\frac{1-i}{2}} \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_A^{\frac{1+i}{2}} \right] \\
&= \text{tr} \left[\sigma_B^{\frac{1+i}{2}} \sigma_B^{-1/2} (\rho_B - \sigma_B) \sigma_B^{-1/2} \sigma_B^{\frac{1-i}{2}} \right] \text{tr} \left[\sigma_A^{\frac{1-i}{2}} \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_A^{\frac{1+i}{2}} \right] \\
&= \text{tr}[\rho_B - \sigma_B] \text{tr}[\rho_A - \sigma_A] \\
&= 0,
\end{aligned}$$

where we have used the fact that states with disjoint supports commute and the factorization of the trace under tensor products.

Therefore,

$$\text{tr} \left[T_B \sigma_{AB}^{\frac{1-i}{2}} T_A \sigma_{AB}^{\frac{1+i}{2}} \right] = \xi_1 + \xi_2.$$

■

7.4.4 STEP 4: HÖLDER'S AND PINSKER'S INEQUALITIES TO OBTAIN A RELATIVE ENTROPY

In the last step of the proof, we bound the error terms obtained in the last step so that we finally obtain a relative entropy between ρ_{AB} and σ_{AB} multiplied by another error term that vanishes whenever σ_{AB} is a tensor product.

Step 7.4.5 With the same notation of the previous steps:

$$\begin{aligned}
&\log \text{tr} M \\
&\leq 2 \left(\int_{-\infty}^{+\infty} dt \beta_0(t) \|S_1(t)\|_{\infty} \left\| \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \sigma_B^{-1/2} \right\|_{\infty} + \int_{-\infty}^{+\infty} dt \beta_0(t) \|S_2(t)\|_{\infty} \right) D(\rho_{AB} \| \sigma_{AB}),
\end{aligned}$$

where $S_1(t)$ and $S_2(t)$ depend only on σ_{AB} and vanish when $\sigma_{AB} = \sigma_A \otimes \sigma_B$.

Proof. Let us bound separately ξ_1 and ξ_2 .

First, we write:

$$S_1(t) := \sigma_{AB}^{\frac{1-i}{2}} - (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}},$$

$$S_2(t) := \sigma_{AB}^{\frac{1+i}{2}} - (\sigma_A \otimes \sigma_B)^{\frac{1+i}{2}},$$

again to simplify notation. Using the submultiplicativity of the Schatten norms, we have for ξ_1

$$\begin{aligned}
\xi_1 &= \int_{-\infty}^{+\infty} dt \beta_0(t) \text{tr} \left[T_B \left(\sigma_{AB}^{\frac{1-i}{2}} - (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} \right) T_A \sigma_{AB}^{\frac{1+i}{2}} \right] \\
&= \int_{-\infty}^{+\infty} dt \beta_0(t) \text{tr} \left[(\rho_B - \sigma_B) \sigma_B^{-1/2} S_1(t) \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \sigma_B^{-1/2} \right] \\
&\leq \|\rho_B - \sigma_B\|_1 \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{-1/2} S_1(t) \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \sigma_B^{-1/2} \right\|_1
\end{aligned}$$

and in virtue of Hölder's inequality,

$$\begin{aligned}\xi_1 &\leq \|\rho_B - \sigma_B\|_1 \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{-1/2} S_1(t) \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \sigma_B^{-1/2} \right\|_1 \\ &\leq \|\rho_B - \sigma_B\|_1 \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{-1/2} S_1(t) \sigma_A^{-1/2} \right\|_\infty \left\| (\rho_A - \sigma_A) \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \sigma_B^{-1/2} \right\|_1 \\ &\leq \|\rho_B - \sigma_B\|_1 \|\rho_A - \sigma_A\|_1 \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{-1/2} S_1(t) \sigma_A^{-1/2} \right\|_\infty \left\| \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \sigma_B^{-1/2} \right\|_\infty.\end{aligned}$$

Now, for the first norm inside the integral, we have

$$\begin{aligned}\left\| \sigma_B^{-1/2} S_1(t) \sigma_A^{-1/2} \right\|_\infty &= \left\| \sigma_B^{-1/2} \left(\sigma_{AB}^{\frac{1-i}{2}} - (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} \right) \sigma_A^{-1/2} \right\|_\infty \\ &= \left\| \sigma_B^{-1/2} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{-1/2} - (\sigma_A \otimes \sigma_B)^{\frac{-i}{2}} \right\|_\infty \\ &= \left\| \sigma_B^{\frac{-1+i}{2}} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{\frac{-1+i}{2}} - \mathbb{1}_{AB} \right\|_\infty,\end{aligned}$$

because of the unitarity invariance of Schatten norms.

Finally, using Pinsker's inequality and the data processing inequality, we have:

$$\|\rho_B - \sigma_B\|_1 \leq \sqrt{2D(\rho_B \| \sigma_B)} \leq \sqrt{2D(\rho_{AB} \| \sigma_{AB})},$$

and analogously for the term with support in A . Thus, we can bound ξ_1 by

$$\xi_1 \leq \left(2 \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{\frac{-1+i}{2}} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{\frac{-1+i}{2}} - \mathbb{1}_{AB} \right\|_\infty \left\| \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \sigma_B^{-1/2} \right\|_\infty \right) D(\rho_{AB} \| \sigma_{AB}).$$

We can do the same for ξ_2 . First,

$$\begin{aligned}\xi_2 &= \int_{-\infty}^{+\infty} dt \beta_0(t) \operatorname{tr} \left[T_B(\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} T_A \left(\sigma_{AB}^{\frac{1+i}{2}} - (\sigma_A \otimes \sigma_B)^{\frac{1+i}{2}} \right) \right] \\ &= \int_{-\infty}^{+\infty} dt \beta_0(t) \operatorname{tr} \left[(\rho_B - \sigma_B) \sigma_B^{-1/2} (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} S_2(t) \sigma_B^{-1/2} \right] \\ &\leq \|\rho_B - \sigma_B\|_1 \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{-1/2} (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} S_2(t) \sigma_B^{-1/2} \right\|_1,\end{aligned}$$

where we have used the submultiplicativity of Schatten norms. Using again Hölder's inequality twice, we can bound this term by:

$$\begin{aligned}\xi_2 &\leq \|\rho_B - \sigma_B\|_1 \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{-1/2} (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} \sigma_A^{-1/2} (\rho_A - \sigma_A) \sigma_A^{-1/2} S_2(t) \sigma_B^{-1/2} \right\|_1 \\ &\leq \|\rho_B - \sigma_B\|_1 \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{-1/2} (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} \sigma_A^{-1/2} \right\|_\infty \left\| (\rho_A - \sigma_A) \sigma_A^{-1/2} S_2(t) \sigma_B^{-1/2} \right\|_1 \\ &\leq \|\rho_B - \sigma_B\|_1 \|\rho_A - \sigma_A\|_1 \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{-1/2} (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} \sigma_A^{-1/2} \right\|_\infty \left\| \sigma_A^{-1/2} S_2(t) \sigma_B^{-1/2} \right\|_\infty\end{aligned}$$

For the first term inside the integral, it is clear that

$$\left\| \sigma_B^{-1/2} (\sigma_A \otimes \sigma_B)^{\frac{1-i}{2}} \sigma_A^{-1/2} \right\|_\infty = 1.$$

Therefore,

$$\xi_2 \leq \|\rho_B - \sigma_B\|_1 \|\rho_A - \sigma_A\|_1 \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_A^{-1/2} S_2(t) \sigma_B^{-1/2} \right\|_\infty,$$

and again

$$\left\| \sigma_A^{-1/2} S_2(t) \sigma_B^{-1/2} \right\|_\infty = \left\| \sigma_B^{\frac{-1-it}{2}} \sigma_{AB}^{\frac{1+it}{2}} \sigma_A^{\frac{-1-it}{2}} - \mathbb{1}_{AB} \right\|_\infty,$$

because of the unitary invariance of Schatten norms.

Finally, as in the case of ξ_1 , in virtue of Pinsker's inequality and the data processing inequality, we obtain:

$$\xi_2 \leq \left(2 \int_{-\infty}^{+\infty} dt \beta_0(t) \left\| \sigma_B^{\frac{-1-it}{2}} \sigma_{AB}^{\frac{1+it}{2}} \sigma_A^{\frac{-1-it}{2}} - \mathbb{1}_{AB} \right\|_\infty \right) D(\rho_{AB} \| \sigma_{AB}).$$

Note that when $\sigma_{AB} = \sigma_A \otimes \sigma_B$, both $S_1(t)$ and $S_2(t)$ vanish, obtaining then an error term for the quasi-factorization result that vanishes when σ_{AB} is a product. ■

R Remark 7.4.6

The result of quasi-factorization obtained in this section presents a much worse error term than the one obtained in the previous section for the conditional relative entropy, in the sense that it might be much more difficult to deal with the former and find examples for which it is actually small.

However, the bounds are clearly not tight. In particular, in the fourth step, we bound ξ_1 and ξ_2 in a very loose way, giving space to possible improvements of the bounds, and, hence, to a possibly better result of quasi-factorization.

R Remark 7.4.7

Similarly to what we mentioned in the previous subsection, Proposition 7.1.3 can be also seen as a particular case of this theorem, when the number of subregions considered is 2. Note again that in the simplification given by the proposition the multiplicative error term disappears, since in that case we were considering σ_{AB} a tensor product.

R Remark 7.4.8

Throughout the proof of the theorem, we are not using strongly enough the fact that we are working with a specific “conditional expectation”, the heat-bath conditional expectation. The application of Lieb's Theorem of course is independent of this fact, but the bound that follows is not. Going back to the beginning of Step 7.4.5, one possible way of defining a more general condition might be the following: From the properties of the conditional expectation, we have

$$\begin{aligned} \text{tr} [\mathbb{E}_B^*(\rho_{AB} - \sigma_{AB}) \mathcal{T}_{\sigma_{AB}} (\mathbb{E}_A^*(\rho_{AB} - \sigma_{AB}))] &= \\ &= \text{tr} [(\mathbb{E}_B^*(\rho_{AB}) - \sigma_{AB}) \mathcal{T}_{\sigma_{AB}} (\mathbb{E}_A^*(\rho_{AB}) - \sigma_{AB})] \\ &= \text{tr} [\mathbb{E}_B^*(\rho_{AB}) \mathcal{T}_{\sigma_{AB}} (\mathbb{E}_A^*(\rho_{AB}))] - \text{tr} [\mathbb{E}_A^*(\rho_{AB})] - \text{tr} [\mathbb{E}_B^*(\rho_{AB})] + \text{tr} [\sigma_{AB}] \\ &= \text{tr} [\mathbb{E}_B^*(\rho_{AB}) \mathcal{T}_{\sigma_{AB}} (\mathbb{E}_A^*(\rho_{AB}))] - 1, \end{aligned}$$

where we have used that $\mathcal{T}_{\sigma_{AB}}$ is self-adjoint with respect to the Hilbert-Schmidt product and $\mathcal{T}_{\sigma_{AB}}(\sigma_{AB}) = \mathbb{1}$. Furthermore, we can also write this term as:

$$\text{tr} [\mathbb{E}_B^*(\rho_{AB} - \sigma_{AB}) \mathcal{T}_{\sigma_{AB}} (\mathbb{E}_A^*(\rho_{AB} - \sigma_{AB}))] = \text{tr} [\mathbb{E}_B^*(\rho_A - \sigma_A) \mathcal{T}_{\sigma_{AB}} (\mathbb{E}_A^*(\rho_B - \sigma_B))],$$

since $\mathbb{E}_A^*(\eta_{AB}) = \mathbb{E}_A^*(\eta_B)$ for every $\eta_{AB} \in \mathcal{S}_{AB}$ and the same holds for \mathbb{E}_B^* . Therefore, we can directly derive that

$$\log \operatorname{tr} M \leq \operatorname{tr}[\mathbb{E}_B^*(\rho_A - \sigma_A) \mathcal{T}_{\sigma_{AB}}(\mathbb{E}_A^*(\rho_B - \sigma_B))],$$

for any conditional expectation. Now let

$$H = \mathbb{E}_B \circ \mathcal{T}_{\sigma_{AB}} \circ \mathbb{E}_A^*,$$

so that $\log \operatorname{tr} M \leq \operatorname{tr}[(\rho_A - \sigma_A)H(\rho_B - \sigma_B)]$. Since we have that

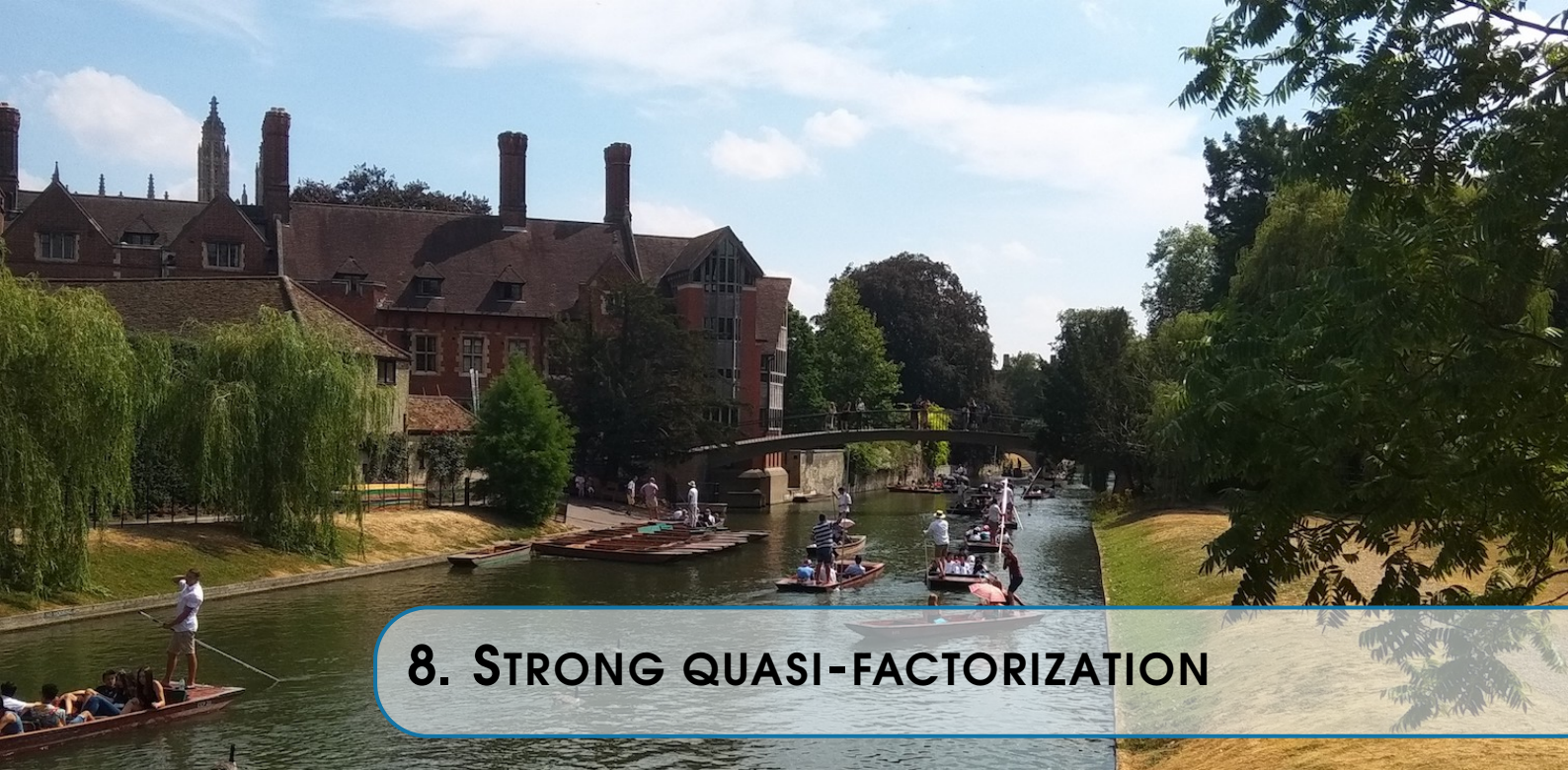
$$\operatorname{tr}[(\rho_A - \sigma_A)\mathbb{1}(\rho_B - \sigma_B)] = \operatorname{tr}[\rho_A - \sigma_A] \operatorname{tr}[\rho_B - \sigma_B] = 0,$$

we can subtract the identity superoperator from the previous bound, and we obtain that the error term is bounded as follows

$$\log \operatorname{tr} M \leq \|H - \mathbb{1}\|_\infty \|\rho_B - \sigma_B\|_1 \|\rho_A - \sigma_A\|_1,$$

obtaining a result which is analogous to Steps 7.4.4 and 7.4.5, which were devoted to bounding $\|H - \mathbb{1}\|_\infty$ in an appropriate way.

However, another completely different approach can be also used for true conditional expectations, due to the properties they present, and allows us to obtain a stronger result of quasi-factorization, as we will show in the next chapter.



8. STRONG QUASI-FACTORIZATION

In the previous chapter, we introduced results of quasi-factorization of the relative entropy as upper bounds for the relative entropy between two states in terms of (at least) two conditional relative entropies in certain subsystems and a multiplicative error term measuring how far the second state is from a tensor product. In the following chapters these results will be used to provide examples of positive log-Sobolev constants for certain dynamics.

However, as we will show in Chapter 10, this family of results only allows for some partial freedom in the geometric recursive argument that one needs to follow in order to lower bound the global log-Sobolev constant in a spin lattice in terms of a conditional one in a subregion of the lattice. That is the main reason for the choice of geometry we use in that chapter, and the reason why we only get the result for the heat-bath dynamics in dimension 1.

In this chapter, we will take a step forward by providing a *strong quasi-factorization of the relative entropy*, namely an upper bound for a *conditional relative entropy* of two density matrices in a subsystem in terms of two conditional relative entropies in certain subsystems of the latter and again a multiplicative error term (see Figure 8.1). A result of this form clearly implies, in particular, a result of quasi-factorization, by restricting the whole lattice to the subsystem in the conditional relative entropy of the smaller part of the inequality.

One could expect that these results constitute a better tool to prove positive log-Sobolev constants. Indeed, as we will see in Chapter 11, the result of strong quasi-factorization of Section 8.4 for the conditional expectation associated to the Davies dynamics will allow us to use a more general geometry than the one mentioned above for the heat-bath dynamics. Indeed, it will be essential to prove a general result of positivity of the log-Sobolev constant for the former.

The counterpart of the approach followed in this chapter is that, for arbitrary states, it only works for general conditional relative entropies by expectations, i.e. when we consider a true conditional expectation instead of the heat-bath one. However, we leave for future work the possibility to extend the results exposed below to that setting, which would directly allow us to extend Chapter 10 to any dimension.

Since the main result presented in this chapter is stronger than its analogues of the previous one, it is reasonable that we have to assume some further condition on the second state than the fact that it is close to be a tensor product. Hence, in Section 8.3, we review this condition and

This is a typical view of the city of Cambridge in a really sunny summer, during *Beyond I.I.D. in information theory 2018*, in July 2018.

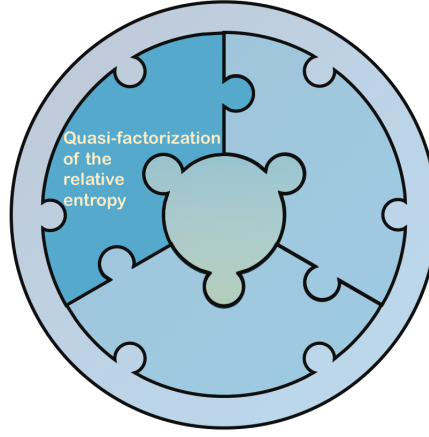


Figure 8.1: Piece associated to the strong quasi-factorization of the relative entropy.

another ones related to it that have appeared previously in the literature of both classical and quantum spin systems, to present subsequently the main result in Section 8.4. However, before that, we show two other examples of strong quasi-factorization under stronger assumptions on the second state, namely the fact that it is a tensor product (obtaining a generalization of Theorem 7.2.1) or a quantum Markov chain (see Section 4.7 for further information on this condition), respectively.

8.1 STRONG QUASI-FACTORIZATION FOR σ A TENSOR PRODUCT

In this section, we begin the presentation of results of strong quasi-factorization of the relative entropy by showing one for the case in which the second state, i.e. σ , is a tensor product.

Note that, even though Theorem 7.2.1 was stated as a quasi-factorization of the relative entropy, a similar proof follows to prove a strong one (see Figure 8.2). More specifically, we can state and prove an upper bound for a conditional relative entropy in a certain region A in terms of the sum of the single-site conditional relative entropies in every site of A .

Theorem 8.1.1 — STRONG QUASI-FACTORIZATION FOR TENSOR PRODUCT.

Let \mathcal{H}_Λ be a multipartite Hilbert space and let $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$ such that $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$. The following inequality holds for every $A \subset \Lambda$:

$$D_A(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{x \in A} D_x(\rho_\Lambda || \sigma_\Lambda). \quad (8.1)$$

Proof. Note that it is enough to prove that for nonempty subregions $A_1, A_2 \subset A$ so that $A_1 \cup A_2 = A$, the following holds

$$D_A(\rho_\Lambda || \sigma_\Lambda) \leq D_{A_1}(\rho_\Lambda || \sigma_\Lambda) + D_{A_2}(\rho_\Lambda || \sigma_\Lambda), \quad (8.2)$$

and proceed inductively.

Let us write $B := \Lambda \setminus A$. Indeed, as we have seen in the proof of Theorem 7.2.1, the terms in

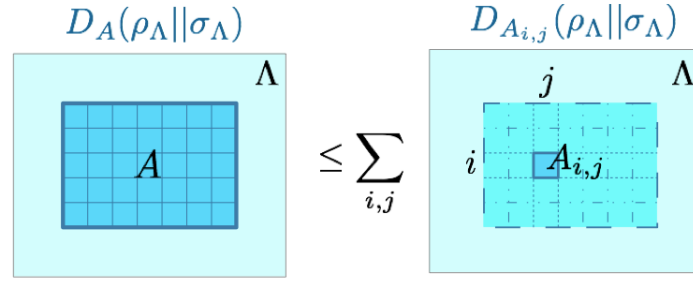


Figure 8.2: Graphical representation for the most general result of strong quasi-factorization under the assumption of σ_Λ tensor product.

σ_Λ cancel out, because of its structure of tensor product. Hence,

$$\begin{aligned}
 D_A(\rho_\Lambda || \sigma_\Lambda) - D_{A_1}(\rho_\Lambda || \sigma_\Lambda) + D_{A_2}(\rho_\Lambda || \sigma_\Lambda) \\
 &= -S(\rho_\Lambda) + S(\rho_B) + S(\rho_\Lambda) - S(\rho_{A_2 B}) + S(\rho_\Lambda) - S(\rho_{A_1 B}) \\
 &= S(\rho_\Lambda) + S(\rho_B) - S(\rho_{A_2 B}) - S(\rho_{A_1 B}) \\
 &\leq 0,
 \end{aligned}$$

where the last inequality follows from the strong subadditivity of the von Neumann entropy. ■

In the next section, we take another step increasing the complexity of these results and assume that σ is not a tensor product, but something close in spirit, a quantum Markov chain.

8.2 STRONG QUASI-FACTORIZATION FOR QUANTUM MARKOV CHAINS

In this section we consider weaker conditions on the second state appearing in the relative entropies than in the previous one and prove another result of (strong) quasi-factorization of the relative entropy in terms of conditional relative entropies (see Figure 8.1). It is strong in the sense that the term appearing in the LHS of the inequality is a conditional relative entropy, although it is weaker than Theorem 8.1.1, because, to prove it, it is necessary that the subregions where we condition are distant enough.

The condition we are assuming now on σ is the fact that it is a quantum Markov chain. We refer the reader to Section 4.7 for further information about the structure of states of this kind.

Theorem 8.2.1 — QUASI-FACTORIZATION FOR QUANTUM MARKOV CHAINS, (Bar+19).

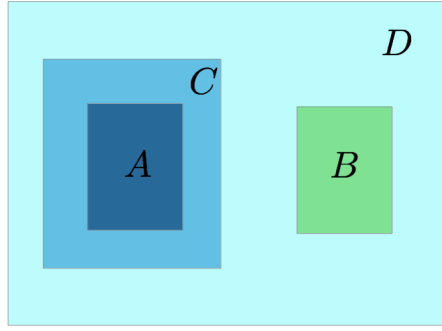
Let $\mathcal{H}_{ABCD} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$ be a 4-partite finite-dimensional Hilbert space, where system C shields A from B and D (see Figure 8.3), and let $\rho_{ABCD}, \sigma_{ABCD} \in \mathcal{S}_{ABCD}$. Let us further assume that σ_{ABCD} is a quantum Markov chain between $A \leftrightarrow C \leftrightarrow BD$. Then, the following inequality holds:

$$D_{AB}(\rho_{ABCD} || \sigma_{ABCD}) \leq D_A(\rho_{ABCD} || \sigma_{ABCD}) + D_B(\rho_{ABCD} || \sigma_{ABCD}). \quad (8.3)$$

Proof. For convenience, we denote $D(\rho_A || \sigma_A)$, respectively $D_A(\rho_{ABCD} || \sigma_{ABCD})$, by $D(A)$, resp. $D_A(ABCD)$, since we are considering the same states ρ_{ABCD} and σ_{ABCD} in every (conditional) relative entropy.

With this notation, it is enough to show:

$$D_{AB}(ABCD) - D_A(ABCD) - D_B(ABCD) \leq 0. \quad (8.4)$$

Figure 8.3: System $ABCD$ where C shields A from BD .

Indeed, it is clear that we have the following:

$$\begin{aligned}
 D_{AB}(ABCD) - D_A(ABCD) - D_B(ABCD) &= D(ABCD) - D(CD) - D(ABCD) + D(BCD) - D(ABCD) + D(ACD) \\
 &= -D(CD) - D(ABCD) + D(BCD) + D(ACD) \\
 &= \text{tr}[\rho_{ABCD}(-\log \rho_{ABCD} - \log \rho_{CD} + \log \rho_{ACD} + \log \rho_{BCD})] \\
 &\quad + \text{tr}[\rho_{ABCD}(\log \sigma_{CD} - \log \sigma_{ACD} + \log \sigma_{ABCD} - \log \sigma_{BCD})] \\
 &= S(\rho_{ABCD}) + S(\rho_{CD}) - S(\rho_{ACD}) - S(\rho_{BCD}) \\
 &\quad + \text{tr}[\rho_{ABCD}(\log \sigma_{CD} - \log \sigma_{ACD} + \log \sigma_{ABCD} - \log \sigma_{BCD})] \\
 &\leq \text{tr}[\rho_{ABCD}(\log \sigma_{CD} - \log \sigma_{ACD} + \log \sigma_{ABCD} - \log \sigma_{BCD})], \tag{8.5}
 \end{aligned}$$

where the last inequality follows from strong subadditivity of the von Neumann entropy. Now, from the structure of quantum Markov chain of the Gibbs state and by Proposition 4.7.5, the sum of logarithms of σ vanishes. ■

This result can be graphically represented as shown in Figure 8.4. A reformulation of this result in terms of quantum spin lattices will be employed in the proof of the positivity of the log-Sobolev constant for the heat-bath dynamics, in Theorem 10.3.3.

Note that, for the proof of the latter theorem, it is necessary that the subregions where we are conditioning are distant enough, and in particular, not adjacent. This is due to the QMC structure of σ and the fact that we are using the “medium” subsystem of the quantum Markov chain (in the formulation of the theorem, subsystem C) to split σ as a direct sum of tensor products and this is essential to cancel the logarithms of σ .

Before introducing the main results of this chapter, which we will do in Section 8.4, let us recall that, in the previous chapter, we were able to prove two results of weak quasi-factorization of the relative entropy for arbitrary states. This will not be the case for results of strong quasi-factorization, for which we will not be able to obtain a general result unless we assume some mild conditions on the second state (which is reasonable, since, in the practice, the result obtained will be much stronger than the ones of the previous chapter, and thus it is normal that we have to pay some price). In the next sections, we study the different conditions of clustering of correlations that we will need to assume for the general result of strong quasi-factorization to hold.

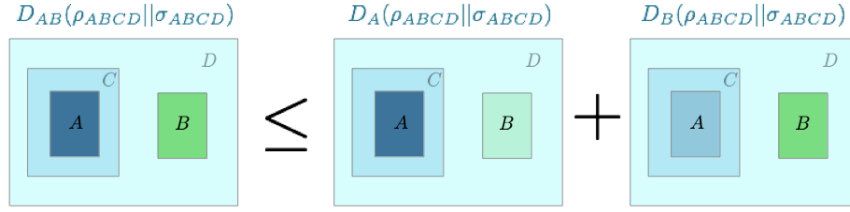


Figure 8.4: Graphical representation of the result of strong quasi-factorization under the assumption of σ_A a quantum Markov chain.

8.3 CLUSTERING OF CORRELATIONS

This section deals with the clustering conditions that we will need to assume in the next part of the thesis to obtain results of positivity for certain logarithmic Sobolev constants (see Figure 8.5). The classical clustering of correlations, which follows from the Dobrushin-Shlosman condition [DS87], is known to be insufficient when dealing with quantum systems. This is due to the possible entanglement at the boundary of the subregion of study. We refer to [KB16] for more details. In order to overcome this issue, the authors of the latter paper introduced the notion of *strong \mathbb{L}_2 -clustering of correlations*. In the classical setting, this notion agrees with the Dobrushin-Shlosman one due to the DLR condition (see Chapter 3).

In this section, we introduce an even stronger notion of conditional clustering, namely the *conditional \mathbb{L}_1 -clustering of correlations*, which will allow us to prove the strong quasi-factorization of the quantum relative entropy. Although we will use it to obtain positivity of a log-Sobolev constant in a spin lattice system, we will state the results of this and the next section for general conditional expectations onto general subalgebras.

More specifically, given a finite-dimensional Hilbert space \mathcal{H} , we will consider three von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$, called \mathcal{N}_1 , \mathcal{N}_2 and \mathcal{M} respectively, such that $\mathcal{N}_1 \cap \mathcal{N}_2 \neq \emptyset$ and, moreover, they satisfy the following *quadrilateral* of inclusions [GJL17]:

$$\begin{pmatrix} \mathcal{N}_1 & \subset & \mathcal{B}(\mathcal{H}) \\ \cup & & \cup \\ \mathcal{M} & \subset & \mathcal{N}_2 \end{pmatrix} \quad (8.6)$$

with corresponding conditional expectations $\mathcal{E}_1 : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{N}_1$, $\mathcal{E}_2 : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{N}_2$ and $\mathcal{E}_{\mathcal{M}} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{M}$, respectively, with respect to a certain state σ .

Before introducing our notion of quantum conditional \mathbb{L}_1 -clustering of correlations, let us recall the concept of strong \mathbb{L}_2 -clustering of correlations introduced in [KB16]. For that, we first need to recall the notion of conditional covariance of two observables. It was introduced in [KB16] as an essential tool for the proof of the positivity of the spectral gap for the heat-bath and Davies dynamics, where it plays the analogous role of the conditional relative entropy in the proof of the positivity of the log-Sobolev constant (see Section 1.2).

We will state it for the particular case of a finite-dimensional Hilbert space associated to a quantum spin lattice, subalgebras of the algebra of bounded operators associated to subregions of the lattice, and the heat-bath and Davies conditional expectations, since it is the original form which appears in the aforementioned paper, although this definition could be extended to a more general framework.

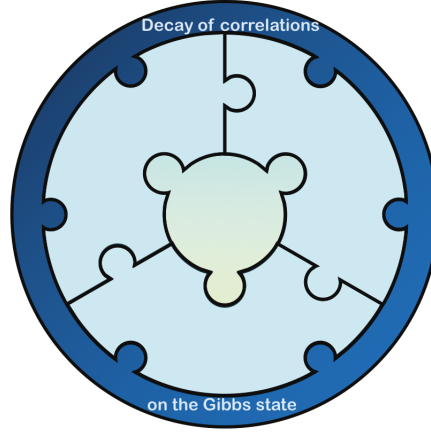


Figure 8.5: Clustering conditions to be assumed on the Gibbs state to prove positivity of the logarithmic Sobolev constant.

Definition 8.3.1 — CONDITIONAL COVARIANCE, (KB16).

Let $\Lambda \subset \subset \mathbb{Z}^d$ be a finite lattice and $A \subseteq \Lambda$. Let $\sigma_\Lambda \in \mathcal{S}_\Lambda$, with $\sigma_\Lambda > 0$, and consider \mathbb{E} to be the “conditional expectation” associated to either the heat-bath or the Davies dynamics. Then, for any $X, Y \in \mathcal{A}_\Lambda$, we define the *conditional covariance* with respect to \mathbb{E} on A by

$$\text{Cov}_{A, \sigma_\Lambda}(X, Y) := |\langle X - \mathbb{E}_A(X), Y - \mathbb{E}_A(Y) \rangle_{\sigma_\Lambda}|,$$

and, similarly, the *conditional variance* with respect to \mathbb{E} on A is defined by $\text{Var}_A(X) := \text{Cov}_A(X, X)$.

Note that the conditional covariance, resp. the conditional variance, reduces to the usual covariance, resp. variance, when $A = \Lambda$. Now, we can state the following condition of clustering of correlations.

Definition 8.3.2 — EXPONENTIAL STRONG \mathbb{L}_2 -CLUSTERING OF CORRELATIONS, (KB16).

Let $\Lambda \subset \subset \mathbb{Z}^d$ be a finite lattice and let $\sigma_\Lambda \in \mathcal{S}_\Lambda$, with $\sigma_\Lambda > 0$. Consider \mathbb{E} to be the “conditional expectation” associated to either the heat-bath or the Davies dynamics. Then, we say that σ_Λ satisfies *exponential strong* (or *conditional*) \mathbb{L}_2 -clustering of correlations if for any $A, B \subset \Lambda$ with $A \cap B \neq \emptyset$, there exist constants $c, \xi > 0$ such that for any $X \in \mathcal{A}_\Lambda$, the following holds:

$$\text{Cov}_{AB, \sigma_\Lambda}(\mathbb{E}_A(X), \mathbb{E}_B(X)) \leq c \|X\|_{\mathbb{L}_2(\sigma_\Lambda)}^2 e^{-d(B \setminus A, A \setminus B)/\xi}. \quad (8.7)$$

Note that we have used a different notation for the conditional expectations of the previous definition to highlight the fact that the result holds for the heat-bath conditional expectation, which, as we discussed in Section 4.3, is not a true conditional expectation.

Moreover, even though we are interested in the use of these clusterings of correlations on the Gibbs state of a certain Hamiltonian, and thus in the context of quantum spin systems, we will introduce them in a more general setting with more general algebras and see how this translates to that setting at the end of this section. Now, we can introduce the following concept.

Definition 8.3.3 — CONDITIONAL \mathbb{L}_1 -CLUSTERING OF CORRELATIONS, (BCR19b).

The state $\sigma \in \mathcal{S}(\mathcal{H})$ is said to satisfy *conditional \mathbb{L}_1 -clustering of correlations* with respect to the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ if there exists a constant $c := c(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M}, \sigma)$ such that, for any $X \in \mathcal{B}(\mathcal{H})$,

$$|\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(X))| \leq c \|X\|_{\mathbb{L}_1(\sigma)}^2. \quad (8.8)$$

Moreover, the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ is said to satisfy *conditional \mathbb{L}_1 -clustering of correlations* if there exists a constant $c = c(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ such that any state $\sigma = \mathcal{E}_{\mathcal{M}}^*(\sigma)$ satisfies conditional \mathbb{L}_1 -clustering of correlations with constant c .

First observe that the conditional \mathbb{L}_1 -clustering implies the following:

Lemma 8.3.4 — (BCR19b).

Assume that the state σ satisfies *conditional \mathbb{L}_1 -clustering of correlations* with respect to the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$. Therefore, for any $X, Y \in \mathcal{B}(\mathcal{H})$,

$$|\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(Y))| \leq (4 + \sqrt{2}) c \max\{\|X\|_{\mathbb{L}_1(\sigma)}^2, \|Y\|_{\mathbb{L}_1(\sigma)}^2\}, \quad (8.9)$$

The above bound can be tightened without loss of generality to the following:

$$|\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(Y))| \leq (4 + \sqrt{2}) c \max\{\|X - \mathcal{E}_{\mathcal{M}}(X)\|_{\mathbb{L}_1(\sigma)}^2, \|Y - \mathcal{E}_{\mathcal{M}}(Y)\|_{\mathbb{L}_1(\sigma)}^2\}.$$

Proof. To simplify the notation, we write $C(Z) := \text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(Z), \mathcal{E}_2(Z))$. Then, the result simply follows from sesquilinearity of the covariance:

$$\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(Y)) = \frac{1}{2} \left(C(X+Y) - iC(X+iY) - (1-i)(C(X) + C(Y)) \right)$$

Then, by Equation (8.8) and the triangle inequality, we have

$$\begin{aligned} C(X+Y) &\leq c \|X+Y\|_{\mathbb{L}_1(\sigma)}^2 \leq 4c \max\{\|X\|_{\mathbb{L}_1(\sigma)}^2, \|Y\|_{\mathbb{L}_1(\sigma)}^2\}, \\ C(X+iY) &\leq 4c \max\{\|X\|_{\mathbb{L}_1(\sigma)}^2, \|Y\|_{\mathbb{L}_1(\sigma)}^2\}, \\ C(X) &\leq c \|X\|_{\mathbb{L}_1(\sigma)}^2, \\ C(Y) &\leq c \|Y\|_{\mathbb{L}_1(\sigma)}^2, \end{aligned}$$

and thus we get

$$|\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(Y))| \leq (4 + \sqrt{2}) c \max\{\|X\|_{\mathbb{L}_1(\sigma)}^2, \|Y\|_{\mathbb{L}_1(\sigma)}^2\}.$$

The second bound follows by a simple centering procedure

$$\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(Y)) = \text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X - \mathcal{E}_{\mathcal{M}}(X)), \mathcal{E}_2(Y - \mathcal{E}_{\mathcal{M}}(Y))).$$

■

The following straightforward lemma provides a simple equivalent definition of \mathbb{L}_1 -clustering that is closer in spirit to the classical strong mixing condition (see Proposition 2.1 of [Ces01]):

Lemma 8.3.5 — (BCR19b).

Let \mathcal{H} be a finite-dimensional Hilbert space and let $\mathcal{N}_1, \mathcal{N}_2, \mathcal{M}$ be von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$ satisfying the quadrilateral of inclusions shown in (8.6). For any $X \in \mathcal{N}_2$, if the following expression holds

$$\|\mathcal{E}_1(X) - \mathcal{E}_{\mathcal{M}}(X)\|_{\infty} \leq c \|X\|_{\mathbb{L}_1(\sigma)}. \quad (8.10)$$

then the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ satisfies conditional \mathbb{L}_1 -clustering of correlations.

Conversely, assume that σ satisfies the conditional \mathbb{L}_1 -clustering of correlations with respect to the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$, with constant c . Then, for any $X \in \mathcal{N}_2$,

$$\|\mathcal{E}_1(X) - \mathcal{E}_{\mathcal{M}}(X)\|_{\infty} \leq (4 + \sqrt{2}) c \max\{1, \|X\|_{\mathbb{L}_1(\sigma)}^2\}. \quad (8.11)$$

Proof. Given any $X \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned} |\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(X))| &= |\langle X, (\mathcal{E}_1 \circ \mathcal{E}_2 - \mathcal{E}_{\mathcal{M}})(X) \rangle_{\sigma}| \\ &\leq \|X\|_{\mathbb{L}_1(\sigma)} \|(\mathcal{E}_1 - \mathcal{E}_{\mathcal{M}}) \circ \mathcal{E}_2(X)\|_{\infty} \\ &\leq c \|X\|_{\mathbb{L}_1(\sigma)} \|\mathcal{E}_2(X)\|_{\mathbb{L}_1(\sigma)} \\ &\leq c \|X\|_{\mathbb{L}_1(\sigma)}^2. \end{aligned}$$

Here, the first line follows from the self-adjointness of \mathcal{E}_1 with respect to $\langle \cdot, \cdot \rangle_{\sigma}$ as well as the fact that $\mathcal{E}_{\mathcal{M}} = \mathcal{E}_1 \circ \mathcal{E}_{\mathcal{M}}$, since $\mathcal{M} \subset \mathcal{N}_1$. The second line arises from Hölder's inequality for weighted \mathbb{L}_p norms, the third line follows from the condition (8.10), and the fourth line from Proposition 4.2.6.

The reverse statement can be proven by duality of weighted \mathbb{L}_p -norms. Indeed, given $X \in \mathcal{N}_2$,

$$\begin{aligned} \|\mathcal{E}_1(X) - \mathcal{E}_{\mathcal{M}}(X)\|_{\infty} &= \sup_{\|Y\|_{\mathbb{L}_1(\sigma)} \leq 1} |\langle Y, \mathcal{E}_1(X) - \mathcal{E}_{\mathcal{M}}(X) \rangle_{\sigma}| \\ &= \sup_{\|Y\|_{\mathbb{L}_1(\sigma)} \leq 1} |\langle \mathcal{E}_1(Y) - \mathcal{E}_{\mathcal{M}}(Y), \mathcal{E}_2(X) - \mathcal{E}_{\mathcal{M}}(X) \rangle_{\sigma}| \\ &\leq (4 + \sqrt{2}) c \sup_{\|Y\|_{\mathbb{L}_1(\sigma)} \leq 1} \max\{\|X\|_{\mathbb{L}_1(\sigma)}^2, \|Y\|_{\mathbb{L}_1(\sigma)}^2\} \\ &= (4 + \sqrt{2}) c \max\{1, \|X\|_{\mathbb{L}_1(\sigma)}^2\}, \end{aligned}$$

where in the first line we use Proposition 4.2.5, the inequality in the third line follows from Lemma 8.3.4, and in the second line we have used the facts that $X \in \mathcal{N}_2$ and $\mathcal{E}_{\mathcal{M}} \circ \mathcal{E}_2 = \mathcal{E}_{\mathcal{M}}$. ■

A conditional expectation can be defined with respect to different invariant states. This is in particular the case of the ones associated to the Davies dynamics, for instance. In the next proposition, we show that the notion of conditional \mathbb{L}_1 -clustering is stable against such a change of invariant state:

Proposition 8.3.6 — (BCR19b).

Assume that $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_{\mathcal{M}}$ are conditional expectations with respect to σ and σ' . If σ' satisfies the conditional \mathbb{L}_1 -clustering with respect to the triplet $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ with constant $c(\sigma')$, then so does σ with constant $c(\sigma) \leq c(\sigma') \|\sigma^{-1/2} \sigma' \sigma^{-1/2}\|_{\infty}$.

Proof. Since $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_{\mathcal{M}}$ are conditional expectations with respect to σ and σ' , and thus,

these states are invariant under them, we have

$$\begin{aligned} \langle (\mathcal{E}_1 - \mathcal{E}_{\mathcal{M}})(X), (\mathcal{E}_2 - \mathcal{E}_{\mathcal{M}})(X) \rangle_{\sigma} &= \langle (\mathcal{E}_1 - \mathcal{E}_{\mathcal{M}})(X), \Gamma_{\sigma'}^{-1} \circ \Gamma_{\sigma} \circ (\mathcal{E}_2 - \mathcal{E}_{\mathcal{M}})(X) \rangle_{\sigma'} \\ &\leq c(\sigma') \|X\|_{\mathbb{L}_1(\sigma')} \|\Gamma_{\sigma'}^{-1} \circ \Gamma_{\sigma}(X)\|_{\mathbb{L}_1(\sigma')} \\ &= c(\sigma') \|X\|_{\mathbb{L}_1(\sigma')} \|X\|_{\mathbb{L}_1(\sigma)}. \end{aligned}$$

The result then follows from the duality between $\mathbb{L}_1(\sigma')$ and $\mathcal{B}(\mathcal{H})$:

$$\begin{aligned} \|X\|_{\mathbb{L}_1(\sigma')} &= \sup_{\|Y\|_{\infty} \leq 1} |\langle Y, X \rangle_{\sigma'}| \\ &= \sup_{\|Y\|_{\infty} \leq 1} |\langle Y, \Gamma_{\sigma'}^{-1} \circ \Gamma_{\sigma}(X) \rangle_{\sigma}| \\ &\leq \sup_{\|Y\|_{\infty} \leq 1} \|\Gamma_{\sigma'} \circ \Gamma_{\sigma}^{-1}(Y)\|_{\infty} \|X\|_{\mathbb{L}_1(\sigma)} \\ &= \|\sigma^{-1/2} \sigma' \sigma^{-1/2}\|_{\infty} \|X\|_{\mathbb{L}_1(\sigma)}, \end{aligned}$$

where the third line follows from Hölder's inequality and the last line follows from the Russo-Dye theorem (see [PT09]). ■

Now, let us study the comparison between two covariances when a subalgebra is contained in another one. The following straightforward lemma shows \mathbb{L}_1 -clustering for a von Neumann algebra \mathcal{N} from \mathbb{L}_1 -clustering for any subalgebra contained in it.

Lemma 8.3.7 — (BCR19b).

For any $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{N}_1 \cap \mathcal{N}_2$, and any $X \in \mathcal{A}(\mathcal{H})$,

$$\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(X)) \geq \text{Cov}_{\mathcal{N}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(X)).$$

Therefore, if the state σ satisfies the *conditional \mathbb{L}_1 -clustering of correlations* with respect to the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$, then it also satisfies it with respect to the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N})$, with constant $c(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}, \sigma) \leq c(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M}, \sigma)$.

Proof. Since the conditional expectations $\mathcal{E}_{\mathcal{N}}$ and $\mathcal{E}_{\mathcal{M}}$ are orthonormal projections with respect to σ , we simply rewrite the conditional covariances as

$$\text{Cov}_{\mathcal{N}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(X)) = \langle \mathcal{E}_1(X), \mathcal{E}_2(X) \rangle_{\sigma} - \langle \mathcal{E}_{\mathcal{N}}(X), \mathcal{E}_{\mathcal{N}}[X] \rangle_{\sigma},$$

and similarly for $\text{Cov}_{\mathcal{M}, \sigma}$. Then, the result follows from

$$\begin{aligned} \langle \mathcal{E}_{\mathcal{M}}(X), \mathcal{E}_{\mathcal{M}}(X) \rangle_{\sigma} &= \langle \mathcal{E}_{\mathcal{M}} \circ \mathcal{E}_{\mathcal{N}}(X), \mathcal{E}_{\mathcal{M}} \circ \mathcal{E}_{\mathcal{N}}(X) \rangle_{\sigma} \\ &\leq \langle \mathcal{E}_{\mathcal{N}}(X), \mathcal{E}_{\mathcal{N}}(X) \rangle_{\sigma}, \end{aligned}$$

where the last inequality follows from the data processing inequality for the $\mathbb{L}_2(\sigma)$ -norm (Proposition 4.2.6), since σ is an invariant state of $\mathcal{E}_{\mathcal{M}}^*$. ■

To conclude this section, let us translate the property of conditional \mathbb{L}_1 -clustering of correlations to the setting of quantum spin systems, and in particular, to the Davies dynamics, introduced in Section 11.1. For that, consider a finite lattice Λ and two overlapping subregions on it, A and B . Then, the conditional expectations of the concepts introduced above are identified in the following form: $\mathcal{E}_1 = \mathcal{E}_A$, $\mathcal{E}_2 = \mathcal{E}_B$, $\mathcal{E}_{\mathcal{N}} = \mathcal{E}_{A \cup B}$ (see Equation (11.9)) and we can write the following condition.

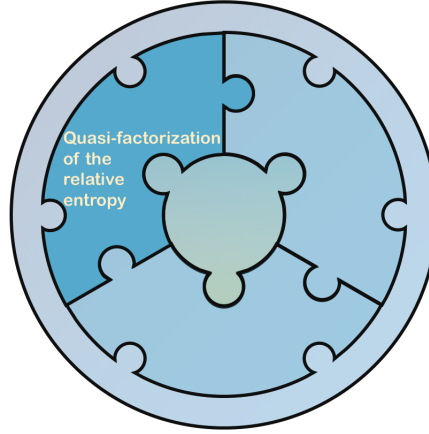


Figure 8.6: Piece associated to the strong quasi-factorization of the relative entropy.

Definition 8.3.8 — EXPONENTIAL CONDITIONAL \mathbb{L}_1 -CLUSTERING OF CORRELATIONS, (BCR19b).

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice. We say that the Davies generator \mathcal{L}_Λ satisfies *exponential conditional \mathbb{L}_1 -clustering of correlations* if there exist constants $c, \xi > 0$ such that, for any $A, B \subset \Lambda$, with $A \cap B \neq \emptyset$, the triple $(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{A \cup B})$ satisfies conditional \mathbb{L}_1 -clustering of correlations with constant $c e^{-d(A \setminus B, B \setminus A)/\xi}$.

In other words, for any $X \in \mathcal{B}(\mathcal{H})$ and any state $\sigma_\Lambda \in \mathcal{S}_\Lambda$, with $\sigma_\Lambda > 0$ and $\sigma_\Lambda = \mathcal{E}_{AB}^*(\sigma_\Lambda)$, the following holds:

$$\text{Cov}_{AB, \sigma_\Lambda}(\mathcal{E}_A(X), \mathcal{E}_B(X)) \leq c \|X\|_{\mathbb{L}_1(\sigma_\Lambda)}^2 e^{-d(A \setminus B, B \setminus A)/\xi}.$$

R Remark 8.3.9

Note that a more general definition could have been introduced for any conditional expectation onto subregions of the original region. However, as we will only use this definition in the setting of Davies dynamics in Chapter 11, we restrict to this definition for simplicity.

In the next section, we will assume that this property of conditional \mathbb{L}_1 -clustering of correlations holds true and we will use it to obtain a strong result of quasi-factorization of the relative entropy for general conditional relative entropies by expectations.

8.4 STRONG QUASI-FACTORIZATION FOR THE GENERAL CONDITIONAL RELATIVE ENTROPY BY EXPECTATIONS

In this section, differently from the results of quasi-factorization of the previous chapter, we consider general conditional relative entropies by expectations. For them, we can prove the following result of strong quasi-factorization of the relative entropy (see Figure 8.6), which essentially differs from those of Chapter 7 in the left-hand side, as now it is also a conditional relative entropy. This stronger result will allow us to overcome some of the issues found in Chapter 10 in the proof of the positive log-Sobolev constant and thus will be fundamental to prove that a positive log-Sobolev constant holds for the Davies dynamics in any dimension. We need to assume the property of conditional \mathbb{L}_1 -clustering of correlations, something that we expect to hold true for many interesting quantum many-body systems (see Chapter 11).

Theorem 8.4.1 — STRONG QUASI-FACTORIZATION UNDER COND. \mathbb{L}_1 -CLUSTERING, (BCR19b).

Let \mathcal{H} be a finite-dimensional Hilbert space and let $\mathcal{N}_1, \mathcal{N}_2, \mathcal{M}$ be von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$ so that $\mathcal{M} \subset \mathcal{N}_1 \cap \mathcal{N}_2$ and they satisfy the quadrilateral of inclusions of (8.6). Let $\mathcal{E}_i : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{N}_i$, for $i = 1, 2$ and $\mathcal{E}_{\mathcal{M}} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{M}$ be conditional expectations with respect to a state σ .

Assume that there exists a constant $0 < c < \frac{1}{2(4+\sqrt{2})}$ such that the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ satisfies the conditional \mathbb{L}_1 -clustering of correlations with corresponding constant c . Then, the following inequality holds for every $\rho \in \mathcal{S}(\mathcal{H})$:

$$D_{\mathcal{M}}^{\mathcal{E}}(\rho||\sigma) \leq \frac{1}{1-2(4+\sqrt{2})c} \left(D_1^{\mathcal{E}}(\rho||\sigma) + D_2^{\mathcal{E}}(\rho||\sigma) \right), \quad (8.12)$$

where $D_{\mathcal{M}}^{\mathcal{E}}(\rho||\sigma) := D(\rho||\mathcal{E}_{\mathcal{M}}^*(\rho))$ and $D_i^{\mathcal{E}}(\rho||\sigma) := D(\rho||\mathcal{E}_i^*(\rho))$ for $i = 1, 2$.

This proof can be split into four steps. The first part of the proof is analogous to that of Theorems 5.0.1 and 7.4.1, but we include it here for the sake of clearness. However, the next steps are different to the ones in the previous theorems in many senses.

8.4.1 STEP 1: ADDITIVE ERROR TERM FOR THE DIFFERENCE OF RELATIVE ENTROPIES

Analogously to Step 7.4.2, we can first prove the following:

Step 8.4.2 In the conditions of Theorem 8.4.1, for every $\rho \in \mathcal{S}(\mathcal{H})$ it holds that

$$D_{\mathcal{M}}^{\mathcal{E}}(\rho||\sigma) \leq D_1^{\mathcal{E}}(\rho||\sigma) + D_2^{\mathcal{E}}(\rho||\sigma) + \log \text{tr} M, \quad (8.13)$$

where $M = \exp \left[-\log \mathcal{E}_{\mathcal{M}}^*(\rho) + \log \mathcal{E}_1^*(\rho) + \log \mathcal{E}_2^*(\rho) \right]$.

Given the conditional expectation of the statement of the theorem, from the definition of general conditional relative entropy by expectations it follows that:

$$\begin{aligned} D_{\mathcal{M}}^{\mathcal{E}}(\rho||\sigma) - D_1^{\mathcal{E}}(\rho||\sigma) - D_2^{\mathcal{E}}(\rho||\sigma) &= D(\rho||\mathcal{E}_{\mathcal{M}}^*(\rho)) - D(\rho||\mathcal{E}_1^*(\rho)) - D(\rho||\mathcal{E}_2^*(\rho)) \\ &= \text{tr} \left[\rho \left(-\log \rho - \underbrace{\log \mathcal{E}_{\mathcal{M}}^*(\rho) + \log \mathcal{E}_1^*(\rho) + \log \mathcal{E}_2^*(\rho)}_{\log M} \right) \right] \\ &= -D(\rho||M). \end{aligned}$$

Moreover, since $\text{tr}[M] \neq 1$ in general, by virtue of Corollary 4.1.9, we have

$$D(\rho||M) \geq -\log \text{tr}[M].$$

8.4.2 STEP 2: ERROR TERM WITH LIEB'S EXTENSION OF GOLDEN-THOMPSON

In the next step, we will bound the error term by the a term in the same spirit than the analogue in Step 8.4.3. We will make use again of Theorem 5.2.1 and Lemma 5.2.3, concerning Lieb's extension of Golden-Thompson inequality and Sutter, Berta and Tomamichel's rotated expression for Lieb's pseudo-inversion operator using multivariate trace inequalities, respectively.

Step 8.4.3 In the conditions above, we have that

$$\log \operatorname{tr} M \leq \int_{-\infty}^{+\infty} dt \beta_0(t) \left\langle X_1 - \mathbb{1}, \Delta_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-it/2} (X_2 - \mathbb{1}) \right\rangle_{\mathcal{E}_{\mathcal{M}}^*(\rho)}, \quad (8.14)$$

with

$$\beta_0(t) = \frac{\pi}{2} (\cosh(\pi t) + 1)^{-1},$$

and

$$X_i := \Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-1/2}(\mathcal{E}_i^*(\rho)) \text{ for } i = 1, 2.$$

Proof. Applying Theorem 5.2.1 to inequality (8.13), we have

$$\begin{aligned} \operatorname{tr} M &= \operatorname{tr} \left[\exp \left(- \underbrace{\log \mathcal{E}_{\mathcal{M}}^*(\rho)}_f + \underbrace{\log \mathcal{E}_1^*(\rho)}_h + \underbrace{\log \mathcal{E}_2^*(\rho)}_g \right) \right] \\ &\leq \operatorname{tr} \left[\mathcal{E}_1^*(\rho) \mathcal{T}_{\mathcal{E}_{\mathcal{M}}^*(\rho)}(\mathcal{E}_2^*(\rho)) \right], \end{aligned}$$

and because of Lemma 5.2.3,

$$\operatorname{tr} M \leq \operatorname{tr} \left[\mathcal{E}_1^*(\rho) \int_{-\infty}^{+\infty} dt \beta_0(t) \mathcal{E}_{\mathcal{M}}^*(\rho)^{-\frac{1-it}{2}} \mathcal{E}_2^*(\rho) \mathcal{E}_{\mathcal{M}}^*(\rho)^{-\frac{1+it}{2}} \right].$$

Now, note that if we subtract $\mathcal{E}_{\mathcal{M}}^*(\rho)$ from $\mathcal{E}_1^*(\rho)$ and $\mathcal{E}_2^*(\rho)$, respectively, we have:

$$\begin{aligned} \operatorname{tr} \left[(\mathcal{E}_1^*(\rho) - \mathcal{E}_{\mathcal{M}}^*(\rho)) \mathcal{E}_{\mathcal{M}}^*(\rho)^{-\frac{1-it}{2}} (\mathcal{E}_2^*(\rho) - \mathcal{E}_{\mathcal{M}}^*(\rho)) \mathcal{E}_{\mathcal{M}}^*(\rho)^{-\frac{1+it}{2}} \right] &= \\ &= \operatorname{tr} \left[\mathcal{E}_1^*(\rho) \mathcal{E}_{\mathcal{M}}^*(\rho)^{-\frac{1-it}{2}} \mathcal{E}_2^*(\rho) \mathcal{E}_{\mathcal{M}}^*(\rho)^{-\frac{1+it}{2}} \right] - 1 - 1 + 1, \end{aligned}$$

since $\mathcal{E}_{\mathcal{M}}^*$, \mathcal{E}_1^* and \mathcal{E}_2^* are conditional expectations and, thus, trace preserving.

Therefore,

$$\begin{aligned} \log \operatorname{tr} M &\leq \log \int_{-\infty}^{+\infty} dt \beta_0(t) \operatorname{tr} \left[\mathcal{E}_1^*(\rho) \mathcal{E}_{\mathcal{M}}^*(\rho)^{-\frac{1-it}{2}} \mathcal{E}_2^*(\rho) \mathcal{E}_{\mathcal{M}}^*(\rho)^{-\frac{1+it}{2}} \right] \\ &= \log \int_{-\infty}^{+\infty} dt \beta_0(t) \left(\operatorname{tr} \left[(\mathcal{E}_1^*(\rho) - \mathcal{E}_{\mathcal{M}}^*(\rho)) \mathcal{E}_{\mathcal{M}}^*(\rho)^{-\frac{1-it}{2}} (\mathcal{E}_2^*(\rho) - \mathcal{E}_{\mathcal{M}}^*(\rho)) \mathcal{E}_{\mathcal{M}}^*(\rho)^{-\frac{1+it}{2}} \right] + 1 \right) \\ &\leq \int_{-\infty}^{+\infty} dt \beta_0(t) \operatorname{tr} \left[(\mathcal{E}_1^*(\rho) - \mathcal{E}_{\mathcal{M}}^*(\rho)) \mathcal{E}_{\mathcal{M}}^*(\rho)^{-\frac{1-it}{2}} (\mathcal{E}_2^*(\rho) - \mathcal{E}_{\mathcal{M}}^*(\rho)) \mathcal{E}_{\mathcal{M}}^*(\rho)^{-\frac{1+it}{2}} \right], \end{aligned}$$

where we have used the following well-known inequality for positive real numbers:

$$\log(x+1) \leq x,$$

and the monotonicity of the logarithm. Finally, if we recall that the Γ operator is given by $\Gamma_{\sigma}^{-1}(\rho) = \sigma^{-1/2} \rho \sigma^{-1/2}$ and we define

$$X_1 := \Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-1}(\mathcal{E}_1^*(\rho)), \quad X_2 := \Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-1}(\mathcal{E}_2^*(\rho)),$$

we can rewrite the previous expression as

$$\begin{aligned}
\log \operatorname{tr} M &\leq \int_{-\infty}^{+\infty} dt \beta_0(t) \operatorname{tr} \left[(\mathcal{E}_1^*(\rho) - \mathcal{E}_{\mathcal{M}}^*(\rho)) \mathcal{E}_{\mathcal{M}}^*(\rho)^{-\frac{1-it}{2}} (\mathcal{E}_1^*(\rho) - \mathcal{E}_{\mathcal{M}}^*(\rho)) \mathcal{E}_{\mathcal{M}}^*(\rho)^{-\frac{1+it}{2}} \right] \\
&= \int_{-\infty}^{+\infty} dt \beta_0(t) \operatorname{tr} \left[(X_1 - \mathbb{1}) \mathcal{E}_{\mathcal{M}}^*(\rho)^{\frac{1-it}{2}} (X_2 - \mathbb{1}) \mathcal{E}_{\mathcal{M}}^*(\rho)^{\frac{1+it}{2}} \right] \\
&= \int_{-\infty}^{+\infty} dt \beta_0(t) \operatorname{tr} \left[(X_1 - \mathbb{1}) \mathcal{E}_{\mathcal{M}}^*(\rho)^{\frac{1}{2}} \Delta_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-it/2} (X_2 - \mathbb{1}) \mathcal{E}_{\mathcal{M}}^*(\rho)^{\frac{1}{2}} \right] \\
&= \int_{-\infty}^{+\infty} dt \beta_0(t) \left\langle X_1 - \mathbb{1}, \Delta_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-it/2} (X_2 - \mathbb{1}) \right\rangle_{\mathcal{E}_{\mathcal{M}}^*(\rho)}.
\end{aligned}$$

■

8.4.3 STEP 3: CONDITIONAL \mathbb{L}_1 -CLUSTERING OF CORRELATIONS TO BOUND THE ERROR TERM

The third step is the one that differs the most from its analogues Step 5.3.2 and Step 7.4.4. Indeed, now we need to make use of the assumption of conditional \mathbb{L}_1 -clustering of correlations to get an upper bound for the error term of the last step in terms of two weighted \mathbb{L}_1 -norms that we will further bound in the final step of the proof.

Step 8.4.4 Assume that there exists a constant $0 < c < \frac{1}{2(4+\sqrt{2})}$ such that the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ satisfies the conditional \mathbb{L}_1 -clustering of correlations with corresponding constant c . Then, we have

$$\begin{aligned}
&\left| \left\langle X_1 - \mathbb{1}, \Delta_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-it/2} (X_2 - \mathbb{1}) \right\rangle_{\mathcal{E}_{\mathcal{M}}^*(\rho)} \right| \\
&\leq (4 + \sqrt{2}) c \max \left\{ \left\| \Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-1}(\rho) - \mathbb{1} \right\|_{\mathbb{L}_1(\mathcal{E}_{\mathcal{M}}^*(\rho))}^2, \left\| \Delta_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-it/2} \left(\Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-1}(\rho) - \mathbb{1} \right) \right\|_{\mathbb{L}_1(\mathcal{E}_{\mathcal{M}}^*(\rho))}^2 \right\}
\end{aligned} \tag{8.15}$$

Proof. First, note that the term in the left-hand side of inequality (8.15) can be written in terms of a conditional covariance. Let us write $X := \Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-1}(\rho)$. It is clear that this notation is consistent with X_1 and X_2 above, i.e., $X_i = \mathcal{E}_i(X)$ for $i = 1, 2$, since

$$\begin{aligned}
X_1 &= \mathcal{E}_{\mathcal{M}}^*(\rho)^{-1/2} \mathcal{E}_1^*(\rho) \mathcal{E}_{\mathcal{M}}^*(\rho)^{-1/2} \\
&= \mathcal{E}_1 \left(\mathcal{E}_{\mathcal{M}}^*(\rho)^{-1/2} \rho \mathcal{E}_{\mathcal{M}}^*(\rho)^{-1/2} \right) \\
&= \mathcal{E}_1(X),
\end{aligned}$$

because $\mathcal{E}_{\mathcal{M}}^*(\rho)$ belongs to the algebra of fixed points of \mathcal{E}_1^* and, thus, the following holds:

$$\Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-1} \circ \mathcal{E}_1 = \mathcal{E}_1^* \circ \Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-1}.$$

Now, on the one side, we have

$$\begin{aligned}
\mathcal{E}_{\mathcal{N}}(X_1) &= \Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-1} \circ \mathcal{E}_{\mathcal{M}}^* \circ \mathcal{E}_1^*(\rho) \\
&= \Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-1} \circ \mathcal{E}_{\mathcal{M}}^*(\rho) \\
&= \mathbb{1},
\end{aligned}$$

whereas, on the other side, analogously we get

$$\begin{aligned}\mathcal{E}_{\mathcal{M}}\left(\Delta_{\mathcal{E}^*(\rho)}^{-it/2} \circ X_2\right) &= \mathcal{E}_{\mathcal{M}}\left(\Delta_{\mathcal{E}^*(\rho)}^{-it/2} \circ \Gamma_{\mathcal{E}^*(\rho)}^{-1} \circ \mathcal{E}_2^*(\rho)\right) \\ &= \Delta_{\mathcal{E}^*(\rho)}^{-it/2} \circ \mathcal{E}_{\mathcal{M}}\left(\Gamma_{\mathcal{E}^*(\rho)}^{-1} \circ \mathcal{E}_2^*(\rho)\right) \\ &= \Delta_{\mathcal{E}^*(\rho)}^{-it/2} \circ \Gamma_{\mathcal{E}^*(\rho)}^{-1} \circ \mathcal{E}_{\mathcal{M}}^*(\rho) \\ &= \mathbb{1}.\end{aligned}$$

Therefore, the left-hand side of Equation (8.15) can be rewritten as

$$\begin{aligned}\left\langle X_1 - \mathbb{1}, \Delta_{\mathcal{E}^*(\rho)}^{-it/2} (X_2 - \mathbb{1}) \right\rangle_{\mathcal{E}^*(\rho)} &= \left\langle (\text{id} - \mathcal{E}_{\mathcal{M}})(X_1), (\text{id} - \mathcal{E}_{\mathcal{M}})\left(\Delta_{\mathcal{E}^*(\rho)}^{-it/2} \circ X_2\right) \right\rangle_{\mathcal{E}^*(\rho)} \\ &= \text{Cov}_{\mathcal{M}, \mathcal{E}^*(\rho)}\left(X_1, \Delta_{\mathcal{E}^*(\rho)}^{-it/2} \circ X_2\right).\end{aligned}$$

Finally, by virtue of Lemma 8.3.4, we get

$$\begin{aligned}&\left| \left\langle X_1 - \mathbb{1}, \Delta_{\mathcal{E}^*(\rho)}^{-it/2} (X_2 - \mathbb{1}) \right\rangle_{\mathcal{E}^*(\rho)} \right| \\ &\leq (4 + \sqrt{2})c \max \left\{ \left\| \Gamma_{\mathcal{E}^*(\rho)}^{-1}(\rho) - \mathbb{1} \right\|_{\mathbb{L}_1(\mathcal{E}^*(\rho))}^2, \left\| \Delta_{\mathcal{E}^*(\rho)}^{-it/2} \left(\Gamma_{\mathcal{E}^*(\rho)}^{-1}(\rho) - \mathbb{1} \right) \right\|_{\mathbb{L}_1(\mathcal{E}^*(\rho))}^2 \right\},\end{aligned}$$

as

$$\mathcal{E}_1\left(\Gamma_{\mathcal{E}^*(\rho)}^{-1}(\rho) - \mathbb{1}\right) = \Gamma_{\mathcal{E}^*(\rho)}^{-1}(\mathcal{E}_1^*(\rho)) - \mathbb{1},$$

and

$$\mathcal{E}_2\left(\Delta_{\mathcal{E}^*(\rho)}^{-it/2} \left(\Gamma_{\mathcal{E}^*(\rho)}^{-1}(\rho) - \mathbb{1} \right)\right) = \Delta_{\mathcal{E}^*(\rho)}^{-it/2} \left(\Gamma_{\mathcal{E}^*(\rho)}^{-1}(\mathcal{E}_2^*(\rho)) - \mathbb{1} \right).$$

■

8.4.4 STEP 4: PROPERTIES OF WEIGHTED \mathbb{L}_p -NORMS TO OBTAIN A RELATIVE ENTROPY

Finally, in the last step of the proof, we use several properties of weighted \mathbb{L}_p -norms to get back a relative entropy.

Step 8.4.5 The following holds for every $\rho \in \mathcal{S}(\mathcal{H})$:

$$\log \text{tr} M \leq 2(4 + \sqrt{2})c D(\rho \| \mathcal{E}^*(\rho)). \quad (8.16)$$

Proof. Let us study separately the two error terms obtained in the previous step. For the first one, it is clear that we have

$$\left\| \Gamma_{\mathcal{E}^*(\rho)}^{-1}(\rho) - \mathbb{1} \right\|_{\mathbb{L}_1(\mathcal{E}^*(\rho))}^2 = \|\rho - \mathcal{E}^*(\rho)\|_1^2,$$

from the definition of the weighted \mathbb{L}_p -norms.

For the second term, by virtue of Proposition 4.2.6, we have

$$\left\| \Delta_{\mathcal{E}^*(\rho)}^{-it/2} \left(\Gamma_{\mathcal{E}^*(\rho)}^{-1}(\rho) - \mathbb{1} \right) \right\|_{\mathbb{L}_1(\mathcal{E}^*(\rho))}^2 \leq \left\| \Gamma_{\mathcal{E}^*(\rho)}^{-1}(\rho) - \mathbb{1} \right\|_{\mathbb{L}_1(\mathcal{E}^*(\rho))}^2,$$

and thus it is upper bounded by the former term.

Using now Pinsker's inequality, we can conclude

$$\begin{aligned}\log \text{tr} M &\leq \left| \text{Cov}_{\mathcal{M}, \mathcal{E}^*(\rho)}\left(\mathcal{E}_1(X), \mathcal{E}_2\left(\Delta_{\mathcal{E}^*(\rho)}^{-it/2}(X)\right)\right) \right| \\ &\leq 2(4 + \sqrt{2})c D(\rho \| \mathcal{E}^*(\rho)).\end{aligned}$$

■

Figure 8.7: Graphical representation of the result of quasi-factorization of Corollary 8.4.6. To simplify the notation, we have made here the following correspondence with the one of the corollary: $\Lambda \mapsto ABCD$, $A \mapsto AB$, $B \mapsto BC$.

As a consequence of Theorem 8.4.1, we can state the following corollary in the context of quantum spin lattices, and more specifically for the Davies dynamics (see Figure 8.7).

Corollary 8.4.6 — STRONG QUASI-FACTORIZATION FOR DAVIES COND. EXPECT., (BCR19b).

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice. Assume that the Davies generator \mathcal{L}_Λ satisfies *exponential conditional \mathbb{L}_1 -clustering of correlations* with constants $c, \xi > 0$ (see Definition 8.3.8). Then, for any $A, B \subset \Lambda$, with $A \cap B \neq \emptyset$, and any $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$, with $\sigma_\Lambda > 0$ and $\sigma_\Lambda = \mathcal{E}_{AB}^*(\sigma_\Lambda)$, the following holds:

$$D_{AB}^{\mathcal{E}}(\rho_\Lambda || \sigma_\Lambda) \leq \frac{1}{1 - 2(4 + \sqrt{2})c e^{-d(A \setminus B, B \setminus A)/\xi}} (D_A^{\mathcal{E}}(\rho_\Lambda || \sigma_\Lambda) + D_B^{\mathcal{E}}(\rho_\Lambda || \sigma_\Lambda)),$$

where $D_{AB}^{\mathcal{E}}(\rho_\Lambda || \sigma_\Lambda) = D(\rho_\Lambda || \mathcal{E}_{AB}^*(\rho_\Lambda))$ and analogously for A and B .

Proof. This corollary clearly follows from the identification of algebras and conditional expectations explained in the previous section and from Theorem 8.4.1 together with Lemma 8.3.4 adapted to Definition 8.3.8 (i.e. with constant $c e^{-d(A \setminus B, B \setminus A)/\xi}$). ■

This result will be of use in Chapter 11, where it will constitute one of the key tools to prove that the Davies dynamics has a positive log-Sobolev constant, under the assumption of exponential conditional \mathbb{L}_1 -clustering of correlations on the Gibbs state of a certain Hamiltonian.

8.5 OTHER CLUSTERING CONDITIONS

In this section, we will compare the notion of conditional \mathbb{L}_1 -clustering of correlations introduced above with some other notions of clustering of correlations.

8.5.1 CONDITIONAL \mathbb{L}_2 -CLUSTERING OF CORRELATIONS

Let us start with a slight generalization to arbitrary finite-dimensional von Neumann algebras of the *conditional \mathbb{L}_2 -clustering of correlations* that was defined in [KB16].

Definition 8.5.1 — CONDITIONAL \mathbb{L}_2 -CLUSTERING OF CORRELATIONS, (KB16).

The state σ is said to satisfy *conditional \mathbb{L}_2 -clustering of correlations* with respect to the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ if there exists a constant $c = c(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M}, \sigma)$ such that, for any $X \in \mathcal{B}(\mathcal{H})$,

$$\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(X)) \leq c \|X\|_{\mathbb{L}_2(\sigma)}^2. \quad (8.17)$$

Note that the only difference with the notion introduced in the current article is the presence of a 2-norm in the right-hand side instead of the 1-norm of the observable appearing in the former. This condition was then shown to imply positivity of the spectral gap for Gibbs samplers of commuting potentials in the aforementioned paper.

It is obvious that conditional \mathbb{L}_1 -clustering implies conditional \mathbb{L}_2 -clustering, since $\|\cdot\|_{\mathbb{L}_1(\sigma)} \leq \|\cdot\|_{\mathbb{L}_2(\sigma)}$. Now, one of the main differences between these two notions is the fact that if a state σ satisfies conditional \mathbb{L}_2 -clustering with respect to the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$, the condition is also satisfied by any other invariant state σ' . To prove this, we need the following technical lemma:

Lemma 8.5.2 — (BCR19b).

Given a conditional expectation $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{N}$ that is invariant with respect to two different full-rank states, ρ and σ , the following holds:

$$\Gamma_\rho^{1/2} \circ \mathcal{E} \circ \Gamma_\rho^{-1/2} = \Gamma_\sigma^{1/2} \circ \mathcal{E} \circ \Gamma_\sigma^{-1/2}$$

Proof. Since we are in finite dimension, the von Neumann algebra \mathcal{N} takes the following form:

$$\mathcal{N} = \bigoplus_i \mathcal{B}(\mathcal{H}_i) \otimes \mathbb{1}_{\mathcal{K}_i},$$

for some decomposition $\mathcal{H} := \bigoplus_i \mathcal{H}_i \otimes \mathcal{K}_i$ of \mathcal{H} . Therefore, since ρ and σ are invariant states of \mathcal{E} , they can be decomposed as follows:

$$\rho = \bigoplus_i \rho_i \otimes \tau_i, \quad \sigma = \bigoplus_i \sigma_i \otimes \tau_i,$$

for given positive definite operators σ_i, ρ_i and where τ_i is given by $\mathbb{1}_{\mathcal{K}_i}/d_{\mathcal{K}_i}$. Hence,

$$\rho^{-1/4} \sigma^{1/4} = \bigoplus_i \rho_i^{-1/4} \sigma_i^{1/4} \otimes \mathbb{1}_{\mathcal{K}_i} \in \mathcal{N}.$$

Then, it is clear that the following string of identities holds for all $Y \in \mathcal{B}(\mathcal{H})$:

$$\begin{aligned} & \rho^{-1/4} \sigma^{1/4} \mathcal{E}(\sigma^{-1/4} \rho^{1/4} Y \rho^{1/4} \sigma^{-1/4}) \sigma^{1/4} \rho^{-1/4} \\ &= \mathcal{E}(\rho^{-1/4} \sigma^{1/4} \sigma^{-1/4} \rho^{1/4} Y \rho^{1/4} \sigma^{-1/4} \sigma^{1/4} \rho^{-1/4}) \\ &= \mathcal{E}(Y). \end{aligned}$$

The result follows after choosing $Y = \rho^{-1/4} X \rho^{-1/4}$. ■

From this lemma, we can prove that the property of conditional \mathbb{L}_2 -clustering of correlations is independent of the invariant state.

Proposition 8.5.3 — (BCR19b).

Let \mathcal{H} be a finite-dimensional Hilbert space and let $\mathcal{N}_1, \mathcal{N}_2$ and \mathcal{M} be von Neumann subalgebras of the algebra $\mathcal{B}(\mathcal{H})$ so that $\mathcal{N}_1 \cap \mathcal{N}_2 \neq \emptyset$ and satisfying the quadrilateral of inclusions of (8.6). Consider a state $\sigma \in \mathcal{S}(\mathcal{H})$. Let $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_{\mathcal{M}}$ be conditional expectations onto $\mathcal{N}_1, \mathcal{N}_2$ and \mathcal{M} , respectively, with respect to σ and assume that the state σ satisfies conditional \mathbb{L}_2 -clustering of correlations with respect to the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ with constant c . Then, if σ' is another invariant state for the three conditional expectations, it also satisfies conditional \mathbb{L}_2 -clustering of correlations with respect to the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ with the same constant.

Proof. Let us show that for any $X \in \mathcal{B}(\mathcal{H})$ the following holds:

$$\sup_{X \in \mathcal{B}(\mathcal{H})} \frac{\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(X))}{\|X\|_{\mathbb{L}_2(\sigma)}^2} = \sup_{X \in \mathcal{B}(\mathcal{H})} \frac{\text{Cov}_{\mathcal{M}, \sigma'}(\mathcal{E}_1(X), \mathcal{E}_2(X))}{\|X\|_{\mathbb{L}_2(\sigma')}^2}.$$

Indeed, if we choose again $Y := \Gamma_{\sigma}^{-1/2}(X)$ and call $X' := \Gamma_{\sigma'}^{-1/2}(Y)$, it is clear that

$$\|X\|_{\mathbb{L}_2(\sigma)}^2 = \|Y\|_2^2 \quad \text{and} \quad \|Y\|_2^2 = \|X'\|_{\mathbb{L}_2(\sigma')}^2.$$

Therefore, we have the following string of identities:

$$\begin{aligned} \sup_{X \in \mathcal{B}(\mathcal{H})} \frac{\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(X))}{\|X\|_{\mathbb{L}_2(\sigma)}^2} &= \sup_{X \in \mathcal{B}(\mathcal{H})} \frac{\langle X, \mathcal{E}_1 \circ \mathcal{E}_2(X) - \mathcal{E}_{\mathcal{M}}(X) \rangle_{\sigma}}{\|X\|_{\mathbb{L}_2(\sigma)}^2} \\ &= \sup_{Y \in \mathcal{B}(\mathcal{H})} \frac{\langle \Gamma_{\sigma}^{-1/2}(X), \mathcal{E}_1 \circ \mathcal{E}_2(\Gamma_{\sigma}^{-1/2}(X)) - \mathcal{E}_{\mathcal{M}}(\Gamma_{\sigma}^{-1/2}(X)) \rangle_{\sigma}}{\|Y\|_2^2} \\ &= \sup_{Y \in \mathcal{B}(\mathcal{H})} \frac{\langle X, \Gamma_{\sigma}^{1/2}(\mathcal{E}_1 \circ \mathcal{E}_2 - \mathcal{E}_{\mathcal{M}})(\Gamma_{\sigma}^{-1/2}(X)) \rangle_{\text{HS}}}{\|Y\|_2^2} \\ &= \sup_{Y \in \mathcal{B}(\mathcal{H})} \frac{\langle \Gamma_{\sigma'}^{-1/2}(X), \mathcal{E}_1 \circ \mathcal{E}_2(\Gamma_{\sigma'}^{-1/2}(X)) - \mathcal{E}_{\mathcal{M}}(\Gamma_{\sigma'}^{-1/2}(X)) \rangle_{\sigma'}}{\|Y\|_2^2} \\ &= \sup_{X' \in \mathcal{B}(\mathcal{H})} \frac{\text{Cov}_{\mathcal{M}, \sigma'}(\mathcal{E}_1(X'), \mathcal{E}_2(X'))}{\|X'\|_{\mathbb{L}_2(\sigma')}^2}, \end{aligned}$$

where we have used Lemma 8.5.2 in the fourth line. ■

R Remark 8.5.4

Let us recall that in Definition 8.3.3 we stated that a triple satisfies conditional \mathbb{L}_1 -clustering of correlations whenever each invariant state for the preduals of the associated conditional expectations satisfies it. From the previous proposition we deduce that an analogous definition for conditional \mathbb{L}_2 -clustering of correlations would be useless, as every invariant state for the preduals of the associated conditional expectations satisfies conditional \mathbb{L}_2 -clustering of correlations as soon as one does.

Moreover, as another consequence of this proposition we realize that the conditions assumed in [KB16] and in the current paper to prove the results of quasitensorization of the variances and the relative entropy, respectively, as key tools for the proof of the positivity of the spectral gap and the log-Sobolev constant, respectively, for the Davies dynamics, are completely analogous, since in the latter they only assumed their condition for one invariant state, but indirectly had it for any invariant state, whereas in our case we directly assume it for every invariant state.

8.5.2 COVARIANCE-ENTROPY CLUSTERING OF CORRELATIONS

To conclude this section, let us introduce another condition of clustering of correlations, which we will further relate with the notion of conditional \mathbb{L}_1 -clustering of correlations and for which we will later present some examples of systems satisfying it.

Definition 8.5.5 — COVARIANCE-ENTROPY CLUSTERING OF CORRELATIONS, (BCR19b).

Let \mathcal{H} be a finite-dimensional Hilbert space and let $\mathcal{N}_1, \mathcal{N}_2$ and \mathcal{M} be von Neumann subalgebras of the algebra $\mathcal{B}(\mathcal{H})$ so that $\mathcal{N}_1 \cap \mathcal{N}_2 \neq \emptyset$ and satisfying the quadrilateral of inclusions of (8.6). Consider a state $\sigma \in \mathcal{S}(\mathcal{H})$. Let $\mathcal{E}_1, \mathcal{E}_2$ and $\mathcal{E}_{\mathcal{M}}$ be conditional expectations onto $\mathcal{N}_1, \mathcal{N}_2$ and \mathcal{M} , respectively, with respect to σ . The state σ is said to satisfy *covariance-entropy clustering of correlations (Cov-RE)* with respect to the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ if there exists a constant $c := c(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M}, \sigma)$ such that, for any $X \in \mathcal{B}(\mathcal{H})$,

$$|\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(X))| \leq c D(\Gamma_{\sigma}(X) \| \Gamma_{\sigma} \circ \mathcal{E}_{\mathcal{M}}(X)). \quad (8.18)$$

Moreover, the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ is said to satisfy *covariance-entropy clustering of correlations* with constant $c = c(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ if any state $\sigma = \mathcal{E}_{\mathcal{M}}^*(\sigma)$ satisfies covariance-entropy clustering of correlations with constant c .

It is obvious from the definition that conditional \mathbb{L}_1 -clustering of correlations with constant c implies covariance-entropy clustering of correlations with constant $(4 + \sqrt{2})c$ by Pinsker's inequality and Lemma 8.3.4. The converse statement is, in general, an open problem, although all these notions agree in the classical setting due to the DLR conditions.

Furthermore, this condition of clustering of correlations can also be shown to imply a result of quasi-factorization such as Theorem 8.4.1. To show that, first we need the following lemma, which could be proven in more generality but here we reduce to a particular case for simplicity.

Lemma 8.5.6 Assume that the state σ satisfies *covariance-entropy clustering of correlations* with respect to the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$. Then, for any $X, Y \in \mathcal{B}(\mathcal{H})$ such that $\text{tr}[\sigma X] = \text{tr}[\sigma Y] = 1$,

$$\begin{aligned} & |\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(Y))| \\ & \leq (4 + \sqrt{2})c \max \{D(\Gamma_{\sigma}(X) \| \Gamma_{\sigma} \circ \mathcal{E}_{\mathcal{M}}(X)), D(\Gamma_{\sigma}(Y) \| \Gamma_{\sigma} \circ \mathcal{E}_{\mathcal{M}}(Y))\}. \end{aligned} \quad (8.19)$$

Proof. To simplify the notation, we write

$$\begin{aligned} C(Z) &:= \text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(Z), \mathcal{E}_2(Z)), \\ D(Z) &:= D(\Gamma_{\sigma}(Z) \| \Gamma_{\sigma} \circ \mathcal{E}_{\mathcal{M}}(Z)). \end{aligned}$$

Then, the result simply follows from the following polarization identity:

$$\text{Cov}_{\mathcal{M}, \sigma}(\mathcal{E}_1(X), \mathcal{E}_2(Y)) = \frac{1}{2} \left(C(X+Y) - iC(X+iY) - (1-i)(C(X) + C(Y)) \right)$$

Indeed, by Equation (8.18), we have

$$\begin{aligned} C(X+Y) &\leq c D(X+Y), \\ C(X+iY) &\leq c D(X+iY), \\ C(X) &\leq c D(X), \\ C(Y) &\leq c D(Y), \end{aligned}$$

and for the first term we can further upper bound it in the following way:

$$\begin{aligned} D(X+Y) &= 2D \left(\frac{\Gamma_{\sigma}(X+Y)}{2} \left\| \frac{\Gamma_{\sigma} \circ \mathcal{E}_{\mathcal{M}}(X+Y)}{2} \right\| \right) \\ &\leq D(\Gamma_{\sigma}(X) \| \Gamma_{\sigma} \circ \mathcal{E}_{\mathcal{M}}(X)) + D(\Gamma_{\sigma}(Y) \| \Gamma_{\sigma} \circ \mathcal{E}_{\mathcal{M}}(Y)) \\ &= D(X) + D(Y), \end{aligned}$$

where we have normalized the first relative entropy and used its property of joint convexity for states. We can proceed analogously with $D(X + iY)$, where we also need to use the fact that the relative entropy is unitary invariant.

Therefore, we can conclude

$$\begin{aligned} & |\text{Cov}_{\mathcal{M},\sigma}(\mathcal{E}_1(X), \mathcal{E}_2(Y))| \\ & \leq (4 + \sqrt{2})c \max \{D(\Gamma_\sigma(X) \parallel \Gamma_\sigma \circ \mathcal{E}_{\mathcal{M}}(X)), D(\Gamma_\sigma(Y) \parallel \Gamma_\sigma \circ \mathcal{E}_{\mathcal{M}}(Y))\}. \end{aligned}$$

■

Theorem 8.5.7 — STRONG QUASI-FACTORIZATION UNDER COV-RE CLUSTERING.

Let \mathcal{H} be a finite-dimensional Hilbert space and let $\mathcal{N}_1, \mathcal{N}_2, \mathcal{M}$ be von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$ so that $\mathcal{M} \subset \mathcal{N}_1 \cap \mathcal{N}_2$ and they satisfy the quadrilateral of inclusions of (8.6). Let $\mathcal{E}_i : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{N}_i$, for $i = 1, 2$ and $\mathcal{E}_{\mathcal{M}} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{M}$ be conditional expectations with respect to a state σ .

Assume that there exists a constant $0 < c < \frac{1}{4 + \sqrt{2}}$ such that the triple $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{M})$ satisfies covariance-entropy clustering of correlations with corresponding constant c . Then, the following inequality holds for every $\rho \in \mathcal{S}(\mathcal{H})$:

$$D_{\mathcal{M}}^{\mathcal{E}}(\rho \parallel \sigma) \leq \frac{1}{1 - (4 + \sqrt{2})c} \left(D_1^{\mathcal{E}}(\rho \parallel \sigma) + D_2^{\mathcal{E}}(\rho \parallel \sigma) \right). \quad (8.20)$$

Proof. The proof of this proposition follows that of Theorem 8.4.1 in the first two steps. Then, we have

$$\begin{aligned} D_{\mathcal{M}}^{\mathcal{E}}(\rho \parallel \sigma) - D_1^{\mathcal{E}}(\rho \parallel \sigma) - D_2^{\mathcal{E}}(\rho \parallel \sigma) & \leq \int_{-\infty}^{+\infty} dt \beta_0(t) \left\langle X_1 - \mathbb{1}, \Delta_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-it/2} (X_2 - \mathbb{1}) \right\rangle_{\mathcal{E}_{\mathcal{M}}^*(\rho)} \\ & = \int_{-\infty}^{+\infty} dt \beta_0(t) \text{Cov}_{\mathcal{M}, \mathcal{E}_{\mathcal{M}}^*(\rho)} \left(\mathcal{E}_1(X), \mathcal{E}_2 \left(\Delta_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-it/2} (X) \right) \right). \end{aligned}$$

Now, by the assumption of covariance-entropy clustering of correlations and Lemma 8.5.6, we have

$$\begin{aligned} & \text{Cov}_{\mathcal{M}, \mathcal{E}_{\mathcal{M}}^*(\rho)} \left(\mathcal{E}_1(X), \mathcal{E}_2 \left(\Delta_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-it/2} (X) \right) \right) \leq \\ & \leq (4 + \sqrt{2})c \max \left\{ D \left(\Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)}(X) \parallel \Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)} \circ \mathcal{E}_{\mathcal{M}}(X) \right), \right. \\ & \quad \left. D \left(\Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)} \left(\Delta_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-it/2} (X) \right) \parallel \Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)} \circ \mathcal{E}_{\mathcal{M}} \left(\Delta_{\mathcal{E}_{\mathcal{M}}^*(\rho)}^{-it/2} (X) \right) \right) \right\}. \end{aligned}$$

Note that both terms considered in the maximum coincide due to unitary invariance of the relative entropy. Moreover, by definition of X , it is clear that $\Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)}(X) = \rho$ and $\Gamma_{\mathcal{E}_{\mathcal{M}}^*(\rho)} \circ \mathcal{E}_{\mathcal{M}}(X) = \mathcal{E}_{\mathcal{M}}^*(\rho)$.

Therefore, we conclude

$$D_{\mathcal{M}}^{\mathcal{E}}(\rho \parallel \sigma) - D_1^{\mathcal{E}}(\rho \parallel \sigma) - D_2^{\mathcal{E}}(\rho \parallel \sigma) \leq (4 + \sqrt{2})c D_{\mathcal{M}}^{\mathcal{E}}(\rho \parallel \sigma).$$

■

The importance of this notion and its associated result of strong quasi-factorization will be shown in Chapter 11, where we will present an example of a physical system verifying this property of clustering of correlations and analyze the positivity of the log-Sobolev constant via results of strong quasi-factorization such as the ones proven above.



SUMMARY OF RESULTS

To conclude Part II, we summarize below all the results of quasi-factorization developed in the previous chapters. Before that, let us recall some concepts of interest that have been introduced in these chapters and will be necessary to classify these results.

Given a tripartite Hilbert space of the form $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, a result of (*weak*) *quasi-factorization of the relative entropy* is an inequality of the form

$$\left[(1 - \xi(\sigma_{ABC})) D(\rho_{ABC} || \sigma_{ABC}) \leq D_{AB}(\rho_{ABC} || \sigma_{ABC}) + D_{BC}(\rho_{ABC} || \sigma_{ABC}) \right] \quad (8.21)$$

for every $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$, D_{AB}, D_{BC} some conditional relative entropies in AB, BC resp. (see Chapter 6), and where $\xi(\sigma_{ABC})$ is an error term that only depends on σ_{ABC} .

Moreover, if we consider instead a 4-partite Hilbert space of the form $\mathcal{H}_{ABCD} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$, a result of *strong quasi-factorization of the relative entropy* takes the form

$$\left[(1 - \xi(\sigma_{ABCD})) D_{ABC}(\rho_{ABCD} || \sigma_{ABCD}) \leq D_{AB}(\rho_{ABCD} || \sigma_{ABCD}) + D_{BC}(\rho_{ABCD} || \sigma_{ABCD}) \right] \quad (8.22)$$

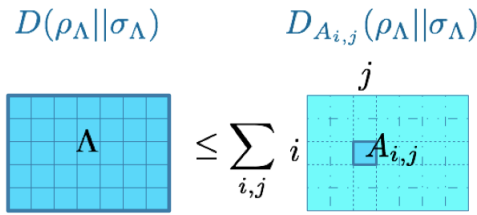
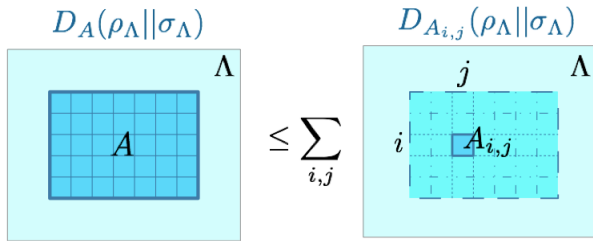
for every $\rho_{ABCD}, \sigma_{ABCD} \in \mathcal{S}_{ABCD}$, D_{ABC}, D_{AB}, D_{BC} some conditional relative entropies in ABC, AB, BC resp., and where $\xi(\sigma_{ABCD})$ is an error term that only depends on σ_{ABCD} .

Let us also recall that we can classify our results of quasi-factorization depending on whether the regions considered in the right hand-side have non-trivial overlap or not. Indeed, for results taking either the form (8.21) or (8.22), we say that they are of the kind (*QF-Ov*), resp. (*QF-NonOv*), if $\dim(\mathcal{H}_B) > 1$, resp. $\dim(\mathcal{H}_B) = 1$.

Taking into account these notions, we can classify all the results of quasi-factorization of the previous chapters in the following table. For each result, we highlight which assumptions are necessary on the states, which notion of conditional relative entropy is employed, whether the result is strong or weak in the sense recalled above, whether it corresponds to a (*QF-Ov*) or (*QF-NonOv*) kind of result and where it appears on the main text.

Typical view of the “Little Mermaid” in Copenhagen with a bike lent by the *QMATH*, center associated to the *University of Copenhagen*, during a visit in May 2018.

ASSUMPTIONS ON ρ, σ	CONDITIONAL R.E.	STRONG/WEAK	(QF-Ov)/(QF-NonOv)	RESULT
$\rho_{A_1, \dots, A_n} = \bigotimes_{i=1}^n \rho_{A_i}$ $\sigma_{A_1, \dots, A_n} = \bigotimes_{i=1}^n \sigma_{A_i}$	CRE and CREexp	Both	Both	Section 7.1
$D_{A_1, \dots, A_j}(\rho_{A_1, \dots, A_n} \sigma_{A_1, \dots, A_n}) \leq \sum_{i=1}^j D_{A_i}(\rho_{A_1, \dots, A_n} \sigma_{A_1, \dots, A_n})$				
ρ_{ABC} arbitrary $\sigma_{ABC} = \sigma_A \otimes \sigma_B \otimes \sigma_C$	CRE and CREexp	Weak	(QF-Ov)	Proposition 7.1.2
$D(\rho_{ABC} \sigma_{ABC}) \leq D_{AB}(\rho_{ABC} \sigma_{ABC}) + D_{BC}(\rho_{ABC} \sigma_{ABC})$				
ρ_{ABC} arbitrary $\sigma_{ABC} = \sigma_A \otimes \sigma_B \otimes \sigma_C$	CRE and CREexp	Weak	(QF-NonOv)	Proposition 7.1.3
$D(\rho_{ABC} \sigma_{ABC}) \leq D_A(\rho_{ABC} \sigma_{ABC}) + D_B(\rho_{ABC} \sigma_{ABC}) + D_C(\rho_{ABC} \sigma_{ABC})$				

ASSUMPTIONS ON ρ, σ	CONDITIONAL R.E.	STRONG/WEAK	(QF-OV)/(QF-NonOV)	RESULT
ρ_Λ arbitrary $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$	CRE and CREexp	Weak	(QF-NonOv)	Theorem 7.2.1
$D(\rho_\Lambda \sigma_\Lambda) \leq \sum_{x \in \Lambda} D_x(\rho_\Lambda \sigma_\Lambda)$				
$D(\rho_\Lambda \sigma_\Lambda) \leq \sum_{i,j} i \cdot D_{A_{i,j}}(\rho_\Lambda \sigma_\Lambda)$ 				
ρ_Λ arbitrary $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$	CRE and CREexp	Strong	(QF-NonOv)	Theorem 8.1.1
$D_A(\rho_\Lambda \sigma_\Lambda) \leq \sum_{x \in A} D_x(\rho_\Lambda \sigma_\Lambda)$				
$D_A(\rho_\Lambda \sigma_\Lambda) \leq \sum_{i,j} i \cdot D_{A_{i,j}}(\rho_\Lambda \sigma_\Lambda)$ 				
ρ_{ABCD} arbitrary σ QMC $A \leftrightarrow C \leftrightarrow BD$	CRE	Strong	(QF-NonOv)	Theorem 8.2.1
$D_{AB}(\rho_{ABCD} \sigma_{ABCD}) \leq D_A(\rho_{ABCD} \sigma_{ABCD}) + D_B(\rho_{ABCD} \sigma_{ABCD})$				
$D_{AB}(\rho_{ABCD} \sigma_{ABCD}) \leq D_A(\rho_{ABCD} \sigma_{ABCD}) + D_B(\rho_{ABCD} \sigma_{ABCD})$ 				

ASSUMPTIONS ON ρ, σ	CONDITIONAL R.E.	STRONG/WEAK	(QF-Ov)/(QF-NonOv)	RESULT
ρ_{ABC} arbitrary σ_{ABC} arbitrary	CRE	Weak	(QF-Ov)	Theorem 7.3.1
$(1 - 2\ H(\sigma_{AC})\ _\infty)D(\rho_{ABC} \sigma_{ABC}) \leq D_{AB}(\rho_{ABC} \sigma_{ABC}) + D_{BC}(\rho_{ABC} \sigma_{ABC})$ $\text{with } H(\sigma_{AC}) = \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \sigma_{AC} \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC}$				
$D(\rho_{ABC} \sigma_{ABC}) \leq \xi\left(\begin{array}{c c c} \sigma_{ABC} \\ A & B & C \end{array}\right) \left(\begin{array}{c c c} D_{AB}(\rho_{ABC} \sigma_{ABC}) \\ A & B & C \end{array} + \begin{array}{c c c} D_{BC}(\rho_{ABC} \sigma_{ABC}) \\ A & B & C \end{array} \right)$				
ρ_{AB} arbitrary σ_{AB} arbitrary	CREexp	Weak	(QF-NonOv)	Theorem 7.4.1
$(1 - \xi(\sigma_{AB})_\infty)D(\rho_{AB} \sigma_{AB}) \leq D_A^E(\rho_{AB} \sigma_{AB}) + D_B^E(\rho_{AB} \sigma_{AB})$ $\text{with } \xi(\sigma_{AB}) = 2(E_1(t) + E_2(t))$ $E_1(t) = \int_{-\infty}^{+\infty} dt \beta_0(t) \left\ \sigma_B^{\frac{-1+i}{2}} \sigma_{AB}^{\frac{1-i}{2}} \sigma_A^{\frac{-1+i}{2}} - \mathbb{1}_{AB} \right\ _\infty \left\ \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+i}{2}} \sigma_B^{-1/2} \right\ _\infty$ $E_2(t) = \int_{-\infty}^{+\infty} dt \beta_0(t) \left\ \sigma_B^{\frac{-1-i}{2}} \sigma_{AB}^{\frac{1+i}{2}} \sigma_A^{\frac{-1-i}{2}} - \mathbb{1}_{AB} \right\ _\infty$				
$D(\rho_{AB} \sigma_{AB}) \leq \xi\left(\begin{array}{c c} \sigma_{AB} \\ A & B \end{array} \leftrightarrow \begin{array}{c c} \sigma_A \otimes \sigma_B \\ A & B \end{array}\right) \left(\begin{array}{c c} D_A^E(\rho_{AB} \sigma_{AB}) \\ A & B \end{array} + \begin{array}{c c} D_B^E(\rho_{AB} \sigma_{AB}) \\ A & B \end{array} \right)$				

ASSUMPTIONS ON ρ, σ	CONDITIONAL R.E.	STRONG/WEAK	(QF-Ov)/(QF-NonOv)	RESULT
ρ_{ABCD} arbitrary σ_{ABCD} arbitrary exp. cond. \mathbb{L}_1 -clust.	gCREexp	Strong	(QF-Ov)	Theorem 8.4.1 Corollary 8.4.6
$\left(1 - 2(4 + \sqrt{2})c e^{-d(A,C)/\xi}\right) D_{ABC}^{\mathcal{E}}(\rho_{ABCD} \sigma_{ABCD}) \leq D_{AB}^{\mathcal{E}}(\rho_{ABCD} \sigma_{ABCD}) + D_{BC}^{\mathcal{E}}(\rho_{ABCD} \sigma_{ABCD})$				

Part III

LOGARITHMIC SOBOLEV INEQUALITIES

The whole puzzle: Logarithmic Sobolev Inequalities

The mixing time of Markovian dissipative evolutions of open quantum many-body systems can be bounded using logarithmic Sobolev constants. As mentioned previously in this text, for classical spin systems the positivity of such constants follows from some mixing conditions on the Gibbs measure, via quasi-factorization results for the entropy.

In the previous part of the thesis, we addressed the problem of finding results of quasi-factorization for the quantum relative entropy. For that, first we had to introduce the notion of quantum conditional relative entropy, which we did in several ways for different quantum dynamics, and we subsequently used those definitions to prove some results of quasi-factorization of the relative entropy.

In this part, we take a further step and address the global problem of proving positivity of log-Sobolev constants for certain quantum systems, following the steps presented in the strategy written in Section 1.2. As we have discussed several times throughout this text, and as we can see in the figure below, this strategy consists of five different steps (or pieces of the puzzle) that we need to ensemble together to obtain the result. The first one of these result-pieces (the three curved pieces of the same colour), namely the quasi-factorization of the relative entropy, was addressed in detail in the previous chapters. We will devote the three chapters of this part to the study of the rest of the pieces in three different settings.

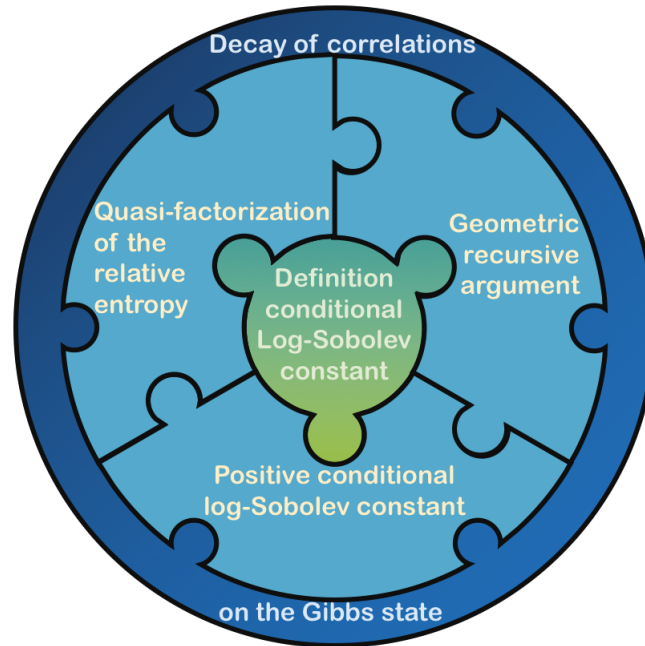


Figure 8.8: Complete puzzle to prove the positivity of a logarithmic Sobolev inequality

In Chapter 9, building on the result of quasi-factorization for tensor products provided in Section 7.2, we study the positivity of the log-Sobolev constant for the heat-bath dynamics with a tensor product fixed point. After providing the suitable definition for the conditional log-Sobolev constant in this setting (from the conditional relative entropy that was employed to prove the aforementioned result of quasi-factorization), we prove the positivity of the log-Sobolev constant

and use a geometric procedure to reduce from the global log-Sobolev constant to the conditional one. In this, the first positive result of our devised strategy, we prove not only that the log-Sobolev constant for this setting is positive, but obtain a lower bound, $1/2$, for its value.

Subsequently, we address the analogous problem for the same heat-bath dynamics imposing weaker conditions of clustering of correlations on its fixed point, which we consider to be the Gibbs state of a local, commuting Hamiltonian. Building on the result of quasi-factorization for the conditional relative entropy provided in Theorem 7.3.1, we devise a new initial geometry on a spin chain where we apply this result and after a geometric recursive argument we manage to reduce the problem of positivity of the global log-Sobolev constant to the conditional one. The conditions of clustering mentioned above, along with some new technical tools, allow us to conclude the positivity of this conditional constant. However, since the result of quasi-factorization on which we build the rest of the proof is “weak” (in the sense that the LHS of the inequality is a global relative entropy instead of a conditional one), a more complicated geometry than in the other cases is necessary, and thus our result only holds in dimension 1.

Finally, we turn to Davies dynamics, where we address the analogous problem, based on the results of strong quasi-factorization from Chapter 8. Considering two different conditions of clustering of correlations, we have proven in the latter chapter two completely analogous results of strong quasi-factorization of the relative entropy, from which now we reduce the positivity of the log-Sobolev constant to the conditional one, as usual (in this setting, using a more standard geometric procedure). Subsequently, we discuss the positivity of the conditional constant and comment on an example satisfying one of the two conditions of clustering of correlations.



9. TENSOR PRODUCT CASE

In this chapter, we will present the first new result derived from this thesis concerning positivity of logarithmic Sobolev constants. More specifically, in the next pages, we will show that the heat-bath dynamics, with product fixed point, has a positive log-Sobolev constant.

The main result of this chapter concerns the heat-bath dynamic, which will be recalled in detail in the next section. For the time being, let us just mention that the global Lindbladian will be defined as the sum of local ones in the following form:

$$\mathcal{L}_\Lambda^* := \sum_{x \in \Lambda} \mathcal{T}_x^* - \text{id}_\Lambda, \quad (9.1)$$

where \mathcal{T}_x^* are certain quantum channels with a fixed point σ_Λ verifying

$$\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x. \quad (9.2)$$

and id_Λ is the superoperator identity acting on \mathcal{B}_Λ .

Therefore, our example constitutes a generalization of a particular case studied in [MFW16b] and [MFW16a], where the authors consider doubly stochastic channels, i.e., channels verifying

$$\mathcal{T}_x^*(\mathbb{1}_\Lambda) = \mathcal{T}_x(\mathbb{1}_\Lambda) = \mathbb{1}_\Lambda,$$

and prove that, if the fixed point is $\sigma_\Lambda = \mathbb{1}_\Lambda / \dim(\Lambda)$, the log-Sobolev constant of a Lindbladian of the form (9.1) is lower bounded by $1/2$ and, hence, positive. Clearly, the identity verifies property (9.2), giving our result more generality in what concerns the fixed point. However, we only manage to prove positivity of the log-Sobolev constant for a certain channel (the Petz recovery map for the partial trace, composed with the partial trace), whereas they obtain it for every channel with the identity as fixed point.

A natural question that arises then is whether one can prove the existence of a positive log-Sobolev constant for a Lindbladian of the form (9.1) for any quantum channel with fixed point satisfying (9.2). That problem is not addressed in this thesis.

Castle of Nagoya, Japan, during the *18th Asian Quantum Information Science Conference (AQIS 2018)*, in September 2018.

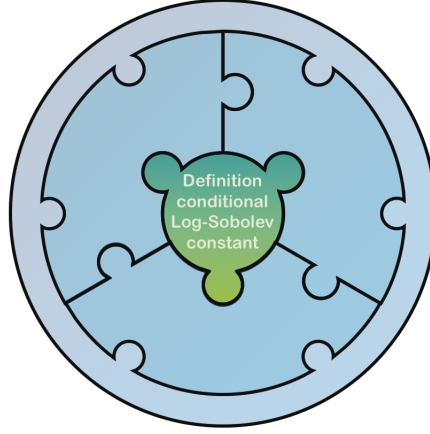


Figure 9.1: Piece for the definition of the conditional log-Sobolev constant

Concerning the positivity of the log-Sobolev for the heat-bath dynamics with tensor product fixed point, this constant was already proven to be positive in [TPK10, Theorem 9]. However, in that result, the authors presented a lower bound for the log-Sobolev constant that depends on some local gaps and the minimum eigenvalues of some local stationary states, whereas the bound that we give in this chapter is universal and independent of any other quantity (indeed, it is $1/2$). Moreover, our proof is completely different and the techniques that we use here are arguably simpler and allow us to show the first example for which the strategy presented in Section 1.2 works. That allows us to think of lifting them to more general examples in quantum many-body systems (as we will do in the following chapters). The main result of this chapter also appeared in [BDR18] in the context of quantum hypothesis testing.

The strategy followed in the proof of this result will be a simplification of that presented in Section 1.2. The five points needed will be the usual ones; some of them, as the conditions to impose on the Gibbs state, quite strong, whereas some other like the geometric recursive argument are pretty simple in this case.

9.1 LOGARITHMIC SOBOLEV INEQUALITY FOR A TENSOR PRODUCT FIXED POINT

The main result of this section is the positivity of the log-Sobolev constant for the heat-bath dynamics with tensor product fixed point. Namely, given $\Lambda \subset \mathbb{Z}^d$ a quantum spin lattice, if we take a product state

$$\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x \quad (9.3)$$

on it, define for every $x \in \Lambda$ the heat-bath conditional expectation with respect to σ_Λ , \mathbb{E}_x^* , as in Section 4.3, and consider the Lindbladian $\mathcal{L}_x^* := \mathbb{E}_x^* - \text{id}_\Lambda$, then the global Lindbladian

$$\mathcal{L}_\Lambda^* := \sum_{x \in \Lambda} \mathcal{L}_x^*$$

is shown to have a positive log-Sobolev constant.

Let us first recall the definition of the heat-bath conditional expectation with respect to σ_Λ :

$$\mathbb{E}_x^*(\rho_\Lambda) := \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2}$$

for all $\rho_\Lambda \in \mathcal{S}_\Lambda$. Since σ_Λ is a product state, we can write $\mathbb{E}_x^*(\rho_\Lambda)$ as

$$\mathbb{E}_x^*(\rho_\Lambda) = \sigma_x \otimes \rho_{x^c}.$$

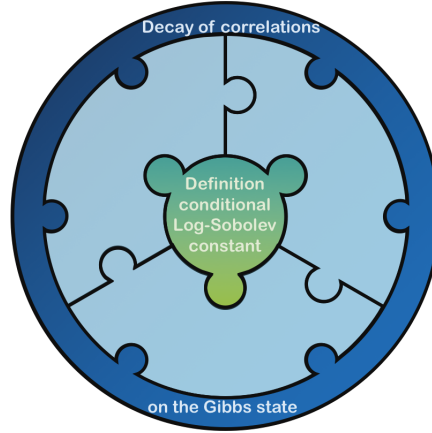


Figure 9.2: Pieces for the definitions of the conditional log-Sobolev constant and decay of correlations on the Gibbs state.

Hence, for every $\rho_\Lambda \in \mathcal{S}_\Lambda$,

$$\mathcal{L}_\Lambda^*(\rho_\Lambda) = \sum_{x \in \Lambda} (\sigma_x \otimes \rho_{x^c} - \rho_\Lambda).$$

Noting the definition of the global Lindbladian as the sum of local ones, one could think on the possibility of reducing the study of a quantity defined on the global Lindbladian to an analogous quantity defined on the Lindbladian associated to every site. Following this idea, we can define specifically a *conditional log-Sobolev constant* (see Figure 9.1), on every subset $A \subset \Lambda$, as an auxiliary quantity for the proof of positivity of the global log-Sobolev constant. We will build the rest of the proof from this definition (as we build the rest of the puzzle starting from this piece).

Definition 9.1.1 — CONDITIONAL LOG-SOBOLEV CONSTANT, (CLP18a).

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice and let $\mathcal{L}_\Lambda^* = \sum_{x \in \Lambda} \mathcal{L}_x^*$ be a global Lindbladian for the Schrödinger picture. Given $A \subset \Lambda$, we define the *conditional log-Sobolev constant* of \mathcal{L}_Λ^* in A by

$$\alpha_\Lambda(\mathcal{L}_A^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\text{tr}[\mathcal{L}_A^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_A(\rho_\Lambda || \sigma_\Lambda)},$$

where σ_Λ is the fixed point of the evolution, and $D_A(\rho_\Lambda || \sigma_\Lambda)$ is the conditional relative entropy.

R Remark 9.1.2

In Section 6.4, we have shown that, when σ_Λ is product, both the conditional relative entropy and the conditional relative entropy by expectations coincide. Since that is the case studied in this subsection, any of them might be the one that appears in the definition of conditional log-Sobolev constant.

Indeed, for every $\rho_\Lambda \in \mathcal{S}_\Lambda$ and $A \subset \Lambda$,

$$D_A(\rho_\Lambda || \sigma_\Lambda) = D_A^E(\rho_\Lambda || \sigma_\Lambda) = I_\rho(A : A^c) + D(\rho_A || \sigma_A).$$

In this case, these definitions also coincide with the one that appears in [Bar17] and [BR18] under the name of *decoherence-free relative entropy*.

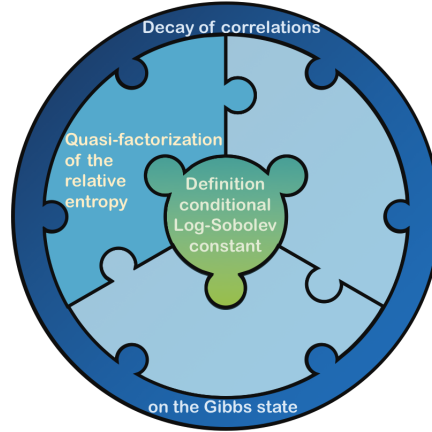


Figure 9.3: Pieces for the definitions of the conditional log-Sobolev constant and decay of correlations on the Gibbs state, and the result of quasi-factorization of the relative entropy. The piece of quasi-factorization appears with well-defined boundaries, as we showed in Section 7.2 that this result, indeed, can be seen as a strong quasi-factorization.

Note that Equation (9.3) provides the condition of decay of correlations imposed on the Gibbs state for the proof to hold (in this case, a very strong assumption). Hence, we have already introduced the two definition-pieces of our puzzle for the log-Sobolev constant (see Figure 9.2).

Now, we can prove the existence of a positive conditional log-Sobolev constant for every local Liouvillian in $x \in \Lambda$, \mathcal{L}_x^* , and use this result to obtain a positive global log-Sobolev constant for \mathcal{L}_Λ^* .

Taking a look at the definition of conditional log-Sobolev constant in $x \in \Lambda$, one can notice that the numerator of the global log-Sobolev constant comes from the sum of the conditional ones. However, the denominators lack a relation of this kind. Therefore, we need the following result of factorization of the relative entropy, which was proven in Section 7.2, to compare both conditional and global log-Sobolev constants via comparing conditional and relative entropies:

Theorem 9.1.3 Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice and let $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$ such that $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$. The following inequality holds:

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda). \quad (9.4)$$

Note that, after this result, we are already in the situation shown in Figure 9.3.

In the following lemma we will prove that the Lindbladian defined at the beginning of this subsection has a positive conditional log-Sobolev constant. Indeed, we will show that this constant can be lower bounded by $1/2$. This, together with the previous result of factorization of the relative entropy, will be later used to prove positivity of the global log-Sobolev constant.

Lemma 9.1.4 — (CLP18a).

For every $x \in \Lambda$ and for \mathcal{L}_x^* defined as above, the following holds:

$$\alpha_\Lambda(\mathcal{L}_x^*) \geq \frac{1}{2}.$$

Proof. Let us write explicitly each term in the definition of $\alpha_\Lambda(\mathcal{L}_x^*)$:

$$\begin{aligned}\alpha_\Lambda(\mathcal{L}_x^*) &= \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\text{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_x(\rho_\Lambda || \sigma_\Lambda)} \\ &= \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{\text{tr}[(\rho_\Lambda - \sigma_x \otimes \rho_{x^c})(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2(D(\rho_\Lambda || \sigma_\Lambda) - D(\rho_{x^c} || \sigma_{x^c}))} \\ &= \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{D(\rho_\Lambda || \sigma_\Lambda) - \text{tr}[\sigma_x \otimes \rho_{x^c}(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2(D(\rho_\Lambda || \sigma_\Lambda) - D(\rho_{x^c} || \sigma_{x^c}))}.\end{aligned}$$

Consider now the second term in the numerator. Since σ_Λ , in particular, splits as a tensor product between the regions x and x^c , we have:

$$\begin{aligned}\text{tr}[\sigma_x \otimes \rho_{x^c}(\log \rho_\Lambda - \log \sigma_\Lambda)] &= \\ &= \text{tr}[\sigma_x \otimes \rho_{x^c}(\log \rho_\Lambda - \log \sigma_x \otimes \rho_{x^c} + \log \sigma_x \otimes \rho_{x^c} - \log \sigma_x \otimes \sigma_{x^c})] \\ &= \text{tr}[\sigma_x \otimes \rho_{x^c}(\log \rho_\Lambda - \log \sigma_x \otimes \rho_{x^c})] + \text{tr}[\rho_{x^c}(\log \rho_{x^c} - \log \sigma_{x^c})] \\ &= -D(\sigma_x \otimes \rho_{x^c} || \rho_\Lambda) + D(\rho_{x^c} || \sigma_{x^c}).\end{aligned}$$

Therefore, $\alpha_\Lambda(\mathcal{L}_x)$ is given by:

$$\begin{aligned}\alpha_\Lambda(\mathcal{L}_x^*) &= \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{D(\rho_\Lambda || \sigma_\Lambda) + D(\sigma_x \otimes \rho_{x^c} || \rho_\Lambda) - D(\rho_{x^c} || \sigma_{x^c})}{2(D(\rho_\Lambda || \sigma_\Lambda) - D(\rho_{x^c} || \sigma_{x^c}))} \\ &= \frac{1}{2} + \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{D(\sigma_x \otimes \rho_{x^c} || \rho_\Lambda)}{2(D(\rho_\Lambda || \sigma_\Lambda) - D(\rho_{x^c} || \sigma_{x^c}))} \\ &\geq \frac{1}{2},\end{aligned}$$

since $D(\rho_\Lambda || \sigma_\Lambda) - D(\rho_{x^c} || \sigma_{x^c}) \geq 0$ (Property of monotonicity of the relative entropy) and $D(\sigma_x \otimes \rho_{x^c} || \rho_\Lambda) \geq 0$ (Property of non-negativity of the relative entropy). ■

This lemma clearly constitutes the piece of the positivity of the log-Sobolev constant. Thus, the situation after having proved it is shown in Figure 9.4.

Finally, we are in position of proving positivity of the global log-Sobolev constant from the previous lemma and Theorem 7.2.1, using the geometric argument, adding thus the last piece to the puzzle.

Theorem 9.1.5 — LOG-SOBOLEV CONSTANT FOR HEAT-BATH FOR TENSOR PRODUCTS, (CLP18a). \mathcal{L}_Λ^* defined as above has a global positive log-Sobolev constant. Moreover, its value is lower bounded by $1/2$.

Proof. In virtue of the result of factorization proven above (Theorem 7.2.1), we know that

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda) \quad (9.5)$$

for every $\rho_\Lambda \in \mathcal{S}_\Lambda$.

From the definition of $\alpha_\Lambda(\mathcal{L}_x^*)$, it is clear that the following holds for every $x \in \Lambda$

$$D_x(\rho_\Lambda || \sigma_\Lambda) \leq \frac{-\text{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{\alpha_\Lambda(\mathcal{L}_x)}.$$

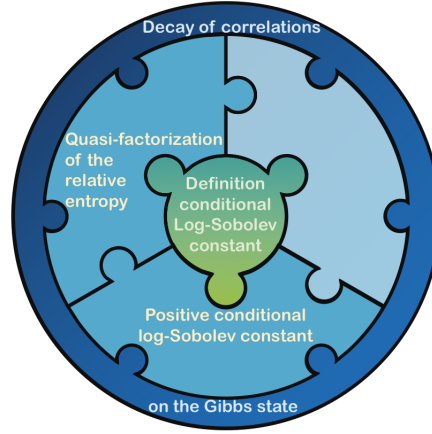


Figure 9.4: Pieces for the definitions of the conditional log-Sobolev constant and decay of correlations on the Gibbs state, and the results of quasi-factorization of the relative entropy and positivity of the conditional log-Sobolev constant.

Putting this together with Equation (9.5), we have:

$$\begin{aligned}
 D(\rho_\Lambda || \sigma_\Lambda) &\leq \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda) \\
 &\leq \sum_{x \in \Lambda} \frac{-\text{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{\alpha_\Lambda(\mathcal{L}_x^*)} \\
 &\leq \frac{1}{\inf_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \sum_{x \in \Lambda} -\text{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \\
 &= \frac{1}{\inf_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} (-\text{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]) \\
 &\leq 2(-\text{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]),
 \end{aligned}$$

where, in the fourth line, we have used the definition of \mathcal{L}_Λ^* and, in the fifth line, Lemma 9.1.4. This expression holds for every $\rho_\Lambda \in \mathcal{S}_\Lambda$.

Finally, recalling the definition of $\alpha(\mathcal{L}_\Lambda^*)$, we have

$$\alpha(\mathcal{L}_\Lambda^*) = \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\text{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)} \geq \frac{1}{2}.$$

Hence, \mathcal{L}_Λ^* has a global positive log-Sobolev constant, which is greater or equal than $1/2$. ■

The proof of Theorem 9.1.5 adds the last piece to the puzzle (see Figure 9.5).

R Remark 9.1.6

Note that, although we use this piece to represent a “geometric recursive argument”, in this case the geometric argument only has one step and thus it is not really recursive. However, we use the notation “recursive”, since in the classical proof whose strategy we are extending here (see Chapter 3), there is indeed a recursion, as well as in some of the examples that appear in the quantum setting in the next chapters.

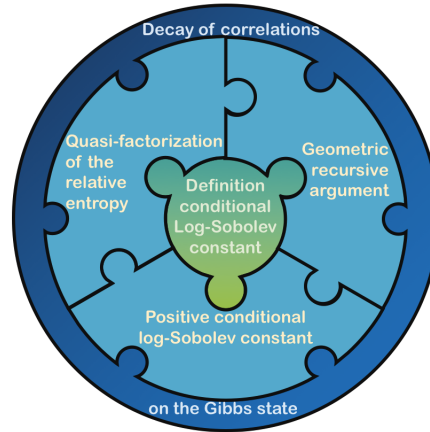


Figure 9.5: Complete puzzle for the proof of positive log-Sobolev constants.

The structure of the proof followed to obtain positivity for the log-Sobolev constant is the first example of the strategy presented in Section 1.2, and this, as we have mentioned in previous chapters, constitutes an analogous quantum version of a simplification to the one used in [DPP02] and [Ces01] to prove a bound on a log-Sobolev constant that connects the decay of correlations in the Gibbs state of a classical spin model to the mixing time of the associated Glauber dynamics. One could then hope of lifting the results of this section to more general situations, that is we could expect that the results of quasi-factorization of the relative entropy of the previous chapters might be of use to obtain positive log-Sobolev constants for certain dynamics and connect it with a decay of correlations on the Gibbs state above the critical temperature. This is addressed in the next two chapters.



10. HEAT-BATH DYNAMICS IN DIMENSION 1

In this chapter we will take a further step in the quest for examples of positive log-Sobolev constants by considering evolutions whose conditions on the fixed points are a bit less restrictive than in the previous one. Namely, we will again consider the heat-bath dynamics and assume that the fixed point of the evolution generated by the heat-bath generator is given by a Gibbs state of a k -local commuting Hamiltonian. For this setting, we will show in the following sections that, under two conditions related to the decay of correlations on the Gibbs state, the Lindbladian has a positive log-Sobolev constant.

As a generalization in some sense of Chapter 9, we will follow the same strategy introduced in Section 1.2, and thus we will split our proof into five steps, two of which consist on finding the proper definitions for certain concepts, whereas the other three constitute three proofs of three different results. However, as opposed to the previous chapter, here all the steps present higher difficulty, especially the last part of the proof, when we need to show positivity of some conditional log-Sobolev constants. To overcome this issue, we introduce a more complicated geometry than the one used in the classical setting in [DPP02], or in the quantum case in [KB16]. This results on the counterpart that our result is only valid for 1D systems. A possible generalization to more dimensions does not seem likely following this approach, but small modifications on it might lead to the desired result. This will be further discussed in Section 10.5 and the Conclusions (see 12.5).

In [KB16], the same setting was considered in arbitrary dimensions and the problem of proving whether the heat-bath generator, under some conditions of clustering of correlations on the fixed point, has a positive spectral gap was addressed and answered positively in the cases where strong clustering of correlations in 2-norm is satisfied. For that, the authors introduced the notion of *conditional spectral gap* and proved positivity of the spectral gap via a result of quasi-factorization of the variance. Here we follow an analogous approach to study the logarithmic Sobolev constant of such system. However, as we will discuss later, the conditions we need to assume here are stronger, the proofs are much longer and more complicated, and our result only holds, so far, in dimension 1.

This is the famous Brandenburger Tor, in Berlin, where I visited the Freie Universität Berlin in March 2019.

10.1 HEAT-BATH DYNAMICS AND CONDITIONAL LOG-SOBOLEV CONSTANT

In this section we will recall the form and basic properties of the heat-bath generator, as well as define the conditional log-Sobolev constant necessary for our strategy.

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice and $\Phi : \Lambda \rightarrow \mathcal{A}_\Lambda$ a k -local bounded commuting potential. Consider σ_Λ to be the associated Gibbs state. Given $A \subseteq \Lambda$, we define the *heat-bath conditional expectation* as follows: For every $\rho_\Lambda \in \mathcal{S}_\Lambda$,

$$\mathbb{E}_A^*(\rho_\Lambda) := \sigma_\Lambda^{1/2} \sigma_{A^c}^{-1/2} \rho_{A^c} \sigma_{A^c}^{-1/2} \sigma_\Lambda^{1/2}.$$

This map has already been introduced in this text in Section 4.3, where we refer the reader for further properties. Let us recall that we can define the *heat-bath generator on Λ* by

$$\mathcal{L}_\Lambda^*(\rho_\Lambda) := \sum_{x \in \Lambda} (\mathbb{E}_x^*(\rho_\Lambda) - \rho_\Lambda),$$

for every $\rho_\Lambda \in \mathcal{S}_\Lambda$. Analogously for every $A \subset \Lambda$, we denote by \mathcal{L}_A^* the generator where the summation is only on elements $x \in A$. Note that the Lindbladian is defined as the sum of terms containing conditional expectations considered over single sites. Some basic properties concerning the heat-bath generator are collected in the following proposition.

Proposition 10.1.1 — (KB16).

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice and $\Phi : \Lambda \rightarrow \mathcal{A}_\Lambda$ a k -local bounded commuting potential. Then, the following properties hold:

1. For any $A \subset \Lambda$, \mathcal{L}_A^* is the generator of a semigroup of CPTP maps of the form $e^{t\mathcal{L}_A^*}$.
2. \mathcal{L}_Λ^* is k -local, in the sense that each individual composing term acts non-trivially only on balls of radius k .
3. For any $A, B \subset \Lambda$, we have

$$\mathcal{L}_A^* + \mathcal{L}_B^* = \mathcal{L}_{A \cup B}^* + \mathcal{L}_{A \cap B}^*.$$

To conclude this subsection, let us introduce, for this Lindbladian, two concepts that will be of use in the proof of the main result, both of them in the line of the conditional relative entropy, since they represent the value of certain notions defined above conditioned to subregions of the whole system. Note that the first one has already been introduced for local Lindbladians in general, although we include here its specific definition for this dynamics for completeness.

Definition 10.1.2 — CONDITIONAL ENTROPY PRODUCTION, (Bar+19).

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice and let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \rightarrow \mathcal{S}_\Lambda$ be the heat-bath generator with fixed point $\sigma_\Lambda \in \mathcal{S}_\Lambda$. Given $A \subset \Lambda$, we define the *conditional entropy production* for every $\rho_\Lambda \in \mathcal{S}_\Lambda$ by

$$\text{EP}_A(\rho_\Lambda) := -\text{tr}[\mathcal{L}_A^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)].$$

Considering the notions of entropy production in a subsystem and conditional relative entropy, one can address again the problem of relating both of them via an inequality, thus obtaining a conditional version of the aforementioned log-Sobolev constant (compare with Definition 9.1.1).

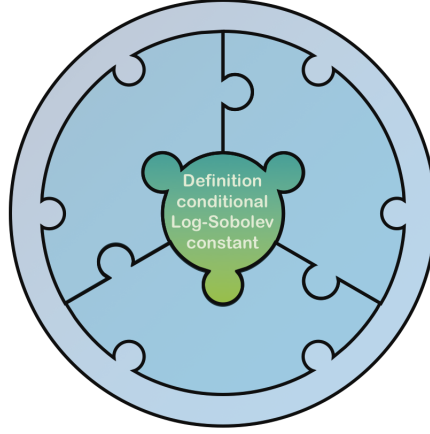


Figure 10.1: Piece for the definition of the conditional log-Sobolev constant

Definition 10.1.3 — CONDITIONAL LOG-SOBOLEV CONSTANT, (Bar+19).

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice and let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \rightarrow \mathcal{S}_\Lambda$ be the heat-bath generator with fixed point $\sigma_\Lambda \in \mathcal{S}_\Lambda$. Given $A \subset \Lambda$, we define the *conditional log-Sobolev constant* of \mathcal{L}_Λ^* by

$$\alpha_\Lambda(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\text{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_A(\rho_\Lambda || \sigma_\Lambda)},$$

where $D_A(\rho_\Lambda || \sigma_\Lambda)$ is the conditional relative entropy introduced in Chapter 6.

In the classical setting, there is no need to define a conditional log-Sobolev constant, since it coincides with the log-Sobolev constant due to the DLR condition [Dob68] [LR69]. Not only this last property fails in general in the quantum case [FW95], but also the study of the conditional log-Sobolev constant is essential in our case, as it is part of our strategy to prove the positivity of the log-Sobolev constant (see Section 1.2 and Figure 10.1).

10.2 TECHNICAL TOOLS

This section aims at presenting a collection of technical results which will be necessary in the proof of the main result of the chapter in Section 10.3. Some of them, as we will see below, are of independent interest to quantum information theory.

The main technical result of these section is Theorem 10.2.5. In its proof, we will make use of the following lemma, which provides a lower bound for a conditional entropy production in a single site (see Definition 10.1.2) in terms of a conditional relative entropy in the same single site.

Lemma 10.2.1 — (Bar+19).

For a single site $x \in \Lambda$, and for every $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$, the following holds

$$\text{EP}_x(\rho_\Lambda) \geq D_x(\rho_\Lambda || \sigma_\Lambda), \quad (10.1)$$

where $\text{EP}_x(\rho_\Lambda)$ is defined with respect to σ_Λ . Therefore, $\text{EP}_A(\rho_\Lambda) \geq 0$ for any $A \subset \Lambda$ and $\rho \in \mathcal{S}_\Lambda$.

Proof. The proof is a direct consequence of the data processing inequality and the fact that $\mathbb{E}_x^*(\cdot)$ is the Petz recovery map for the partial trace in x , composed with the partial trace (and, in

particular, a quantum channel). Indeed, let us recall that $\text{EP}_x(\rho_\Lambda)$ is given by

$$\begin{aligned}\text{EP}_x(\rho_\Lambda) &= -\text{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \\ &= \text{tr}[(\rho_\Lambda - \mathbb{E}_x^*(\rho_\Lambda))(\log \rho_\Lambda - \log \sigma_\Lambda)] \\ &= D(\rho_\Lambda || \sigma_\Lambda) - \text{tr}[\mathbb{E}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)].\end{aligned}\quad (10.2)$$

In the second term of (10.2), let us add and subtract $\log \mathbb{E}_x^*(\rho_\Lambda)$. Then,

$$\begin{aligned}\text{tr}[\mathbb{E}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] &= \text{tr}[\mathbb{E}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda + \log \mathbb{E}_x^*(\rho_\Lambda) - \log \mathbb{E}_x^*(\rho_\Lambda))] \\ &= -D(\mathbb{E}_x^*(\rho_\Lambda) || \rho_\Lambda) + D(\mathbb{E}_x^*(\rho_\Lambda) || \sigma_\Lambda) \\ &\leq D(\mathbb{E}_x^*(\rho_\Lambda) || \sigma_\Lambda),\end{aligned}\quad (10.3)$$

where we have used the fact that the relative entropy of two states is always non-negative.

Finally, since $\mathbb{E}_x^*(\cdot)$ is the Petz recovery map for the partial trace in x composed with the partial trace (denote $\mathbb{E}_x^*(\cdot) = \mathcal{R}_{x^c \rightarrow \Lambda}^\sigma \circ \text{tr}_x[\cdot]$), note that σ_Λ is a fixed point. Then,

$$\begin{aligned}D(\mathbb{E}_x^*(\rho_\Lambda) || \sigma_\Lambda) &= D(\mathcal{R}_{x^c \rightarrow \Lambda}^\sigma \circ \text{tr}_x[\rho_\Lambda] || \mathcal{R}_{x^c \rightarrow \Lambda}^\sigma \circ \text{tr}_x[\sigma_\Lambda]) \\ &\leq D(\rho_{x^c} || \sigma_{x^c}),\end{aligned}$$

and thus

$$\text{EP}_x(\rho_\Lambda) \geq D(\rho_\Lambda || \sigma_\Lambda) - D(\rho_{x^c} || \sigma_{x^c}) = D_x(\rho_\Lambda || \sigma_\Lambda).$$

■

R Remark 10.2.2

If we recall the definition for conditional log-Sobolev introduced in the previous section, Lemma 10.2.1 can clearly be seen as a lower bound for the conditional log-Sobolev constant in a single site $x \in \Lambda$ for the heat-bath dynamics, i.e.

$$\alpha_\Lambda(\mathcal{L}_x^*) \geq \frac{1}{2}.$$

This inequality, in particular, can be used to prove positivity of the log-Sobolev constant for the heat-bath dynamics when σ_Λ is a tensor product, as we showed in Chapter 9 (see also [BDR18] and [Bar17]).

R Remark 10.2.3

Note that, in the previous lemma, we have only used the fact that the partial trace is a quantum channel and $\mathbb{E}_x^*(\cdot)$ its Petz recovery map composed with it. Hence, in more generality, Lemma 10.2.1 could be stated as: Let \mathcal{T} be a quantum channel and denote by $\widehat{\mathcal{T}}$ its Petz recovery map with respect to σ_Λ . Then, for any $\rho_\Lambda \in \mathcal{S}_\Lambda$ the following holds

$$\text{tr}[(\rho_\Lambda - \widehat{\mathcal{T}} \circ \mathcal{T}(\rho_\Lambda))(\log \rho_\Lambda - \log \sigma_\Lambda)] \geq D(\rho_\Lambda || \sigma_\Lambda) - D(\mathcal{T}(\rho_\Lambda) || \mathcal{T}(\sigma_\Lambda)).$$

Another tool that will be of use in the main result of this section is the following lemma, which appeared first in [MOZ98]. It can be seen as an equivalence between blocks of spins, and allows us to prove an equivalence between the usual conditional Lindbladian associated to the heat-bath dynamics in $A \subseteq \Lambda$, given as a sum of local terms, and a modified one given as a unique term. Note that it is stated in the Heisenberg picture.

Lemma 10.2.4 — EQUIVALENCE OF BLOCKS, (MOZ98).

Let $A \subseteq \Lambda$ and let σ_Λ be the Gibbs state of the k -local commuting Hamiltonian mentioned above. There exist constants $0 < c_A, C_A < \infty$, possibly depending on A but not on Λ , such that for any $f_\Lambda \in \mathcal{A}_\Lambda$ the following holds:

$$c_A \sum_{x \in A} \langle f_\Lambda, f_\Lambda - \mathbb{E}_x(f_\Lambda) \rangle_{\sigma_\Lambda} \leq \langle f_\Lambda, f_\Lambda - \mathbb{E}_A(f_\Lambda) \rangle_{\sigma_\Lambda} \leq C_A \sum_{x \in A} \langle f_\Lambda, f_\Lambda - \mathbb{E}_x(f_\Lambda) \rangle_{\sigma_\Lambda}, \quad (10.4)$$

where \mathbb{E}_x , resp. \mathbb{E}_A , is the dual of \mathbb{E}_x^* , resp. of \mathbb{E}_A^* , and is given by

$$\mathbb{E}_x(f_\Lambda) := \sigma_{x^c}^{-1/2} \text{tr}_x[\sigma_\Lambda^{1/2} f_\Lambda \sigma_\Lambda^{1/2}] \sigma_{x^c}^{-1/2},$$

for every $f_\Lambda \in \mathcal{A}_\Lambda$ and analogously for \mathbb{E}_A .

Let us now state and prove the main technical result of this section, which will be essential for the proof of Theorem 10.3.3, but has independent interest on its own.

Theorem 10.2.5 — EQUIVALENCE OF RECOVERY, (Bar+19).

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice and let $\sigma_\Lambda \in \mathcal{S}_\Lambda$ be the Gibbs state of a commuting Hamiltonian over Λ . For any $A \subseteq \Lambda$ and $\rho_\Lambda \in \mathcal{S}_\Lambda$, the following equivalence holds:

$$\rho_\Lambda = \mathbb{E}_A^*(\rho_\Lambda) \Leftrightarrow \rho_\Lambda = \mathbb{E}_x^*(\rho_\Lambda) \quad \forall x \in A. \quad (10.5)$$

Proof. Let us first recall that, for every $\rho_\Lambda \in \mathcal{S}_\Lambda$, the local Lindbladian in $A \subseteq \Lambda$ is given by

$$\mathcal{L}_A^*(\rho_\Lambda) = \sum_{x \in A} (\mathbb{E}_x^*(\rho_\Lambda) - \rho_\Lambda),$$

and define

$$\widetilde{\mathcal{L}}_A^*(\rho_\Lambda) := \mathbb{E}_A^*(\rho_\Lambda) - \rho_\Lambda.$$

Analogously, defining the superoperator $\Gamma_{\sigma_\Lambda} : f_\Lambda \mapsto \sigma_\Lambda^{1/2} f_\Lambda \sigma_\Lambda^{1/2}$, we can write every observable $f_\Lambda \in \mathcal{A}_\Lambda$ as

$$f_\Lambda = \Gamma_{\sigma_\Lambda}^{-1}(\rho_\Lambda) = \sigma_\Lambda^{-1/2} \rho_\Lambda \sigma_\Lambda^{-1/2}, \quad (10.6)$$

and thus we have

$$\mathcal{L}_A(f_\Lambda) = \sum_{x \in A} (\mathbb{E}_x(f_\Lambda) - f_\Lambda),$$

$$\widetilde{\mathcal{L}}_A(f_\Lambda) = \mathbb{E}_A(f_\Lambda) - f_\Lambda.$$

With this notation, inequality (10.4) in Lemma 10.2.4 can be rewritten as

$$-c_A \langle f_\Lambda, \mathcal{L}_A(f_\Lambda) \rangle_{\sigma_\Lambda} \leq -\langle f_\Lambda, \widetilde{\mathcal{L}}_A(f_\Lambda) \rangle_{\sigma_\Lambda} \leq -C_A \langle f_\Lambda, \mathcal{L}_A(f_\Lambda) \rangle_{\sigma_\Lambda},$$

and thus,

$$\langle f_\Lambda, \mathcal{L}_A(f_\Lambda) \rangle_{\sigma_\Lambda} = 0 \quad \Leftrightarrow \quad \forall x \in A, \quad \langle f_\Lambda, \mathcal{L}_x(f_\Lambda) \rangle_{\sigma_\Lambda} = 0 \quad \Leftrightarrow \quad \langle f_\Lambda, \widetilde{\mathcal{L}}_A(f_\Lambda) \rangle_{\sigma_\Lambda} = 0,$$

which thanks to the detailed-balance condition leads to

$$\mathcal{L}_A(f_\Lambda) = 0 \quad \Leftrightarrow \quad \forall x \in A, \quad \mathcal{L}_x(f_\Lambda) = 0 \quad \Leftrightarrow \quad \widetilde{\mathcal{L}}_A(f_\Lambda) = 0. \quad (10.7)$$

Now, because of (10.6), one can easily see that $\mathbb{E}_x^* = \Gamma_{\sigma_\Lambda} \circ \mathbb{E}_x \circ \Gamma_{\sigma_\Lambda}^{-1}$ (see Section 4.3) and the same holds for \mathbb{E}_A^* . Hence, (10.7) is equivalent to

$$\mathcal{L}_A^*(\rho_\Lambda) = 0 \quad \Leftrightarrow \quad \forall x \in A, \quad \mathcal{L}_x^*(\rho_\Lambda) = 0 \quad \Leftrightarrow \quad \widetilde{\mathcal{L}}_A^*(\rho_\Lambda) = 0. \quad (10.8)$$

Recalling the expressions for $\mathcal{L}_A^*(\rho_\Lambda)$ and $\widetilde{\mathcal{L}}_A^*(\rho_\Lambda)$, we obtain:

$$\rho_\Lambda = \mathbb{E}_A^*(\rho_\Lambda) \Leftrightarrow \rho_\Lambda = \mathbb{E}_x^*(\rho_\Lambda) \quad \forall x \in A.$$

■

This result can also be stated in terms of conditional relative entropies. Indeed, note that, as a consequence of Petz's characterization for conditions of equality in the data processing inequality, all the conditions above can be seen as necessary and sufficient conditions for vanishing conditional relative entropies. We have then the following corollary.

Corollary 10.2.6 — (Bar+19).

Let $\Lambda \subset \subset \mathbb{Z}^d$ be a finite quantum lattice and let $\sigma_\Lambda \in \mathcal{S}_\Lambda$ be the Gibbs state of a commuting Hamiltonian. For any $A \subseteq \Lambda$ and $\rho_\Lambda \in \mathcal{S}_\Lambda$, the following equivalence holds:

$$D_A(\rho_\Lambda || \sigma_\Lambda) = 0 \Leftrightarrow D_x(\rho_\Lambda || \sigma_\Lambda) = 0 \quad \forall x \in A. \quad (10.9)$$

Another consequence of the previous result is that a state is recoverable from a certain region whenever it is recoverable from several components of that region that cover it completely, no matter the size of those components. More specifically, we have the following corollary.

Corollary 10.2.7 — (Bar+19).

Given a finite lattice Λ , a partition of it into three subregions A, B, C , and σ_{ABC} the Gibbs state of a commuting Hamiltonian, if we denote by $\mathbb{E}_A^*(\cdot)$ the conditional expectation on A associated to the heat-bath dynamics (with respect to the Gibbs state), we have for any $\rho_{ABC} \in \mathcal{S}_{ABC}$:

$$\mathbb{E}_{AB}^*(\rho_{ABC}) = \rho_{ABC} \Leftrightarrow \begin{cases} \mathbb{E}_A^*(\rho_{ABC}) = \rho_{ABC} \\ \wedge \\ \mathbb{E}_B^*(\rho_{ABC}) = \rho_{ABC}. \end{cases} \quad (10.10)$$

In particular,

$$D_{AB}(\rho_{ABC} || \sigma_{ABC}) = 0 \Leftrightarrow \begin{cases} D_A(\rho_{ABC} || \sigma_{ABC}) = 0 \\ \wedge \\ D_B(\rho_{ABC} || \sigma_{ABC}) = 0. \end{cases} \quad (10.11)$$

Proof. By virtue of Theorem 10.2.5, it is clear that

$$\begin{aligned} \mathbb{E}_{AB}^*(\rho_{ABC}) = \rho_{ABC} &\Leftrightarrow \mathbb{E}_x^*(\rho_{ABC}) = \rho_{ABC} \quad \forall x \in A \cup B \\ &\Leftrightarrow \begin{cases} \mathbb{E}_x^*(\rho_{ABC}) = \rho_{ABC} \quad \forall x \in A \\ \wedge \\ \mathbb{E}_x^*(\rho_{ABC}) = \rho_{ABC} \quad \forall x \in B \end{cases} \Leftrightarrow \begin{cases} \mathbb{E}_A^*(\rho_{ABC}) = \rho_{ABC} \\ \wedge \\ \mathbb{E}_B^*(\rho_{ABC}) = \rho_{ABC} \end{cases} \end{aligned}$$

The second part is a direct consequence of [Pet86] and Corollary 10.2.6. ■

In the next section, we present the main result of this chapter, where all the technical tools presented in this section will be of use.

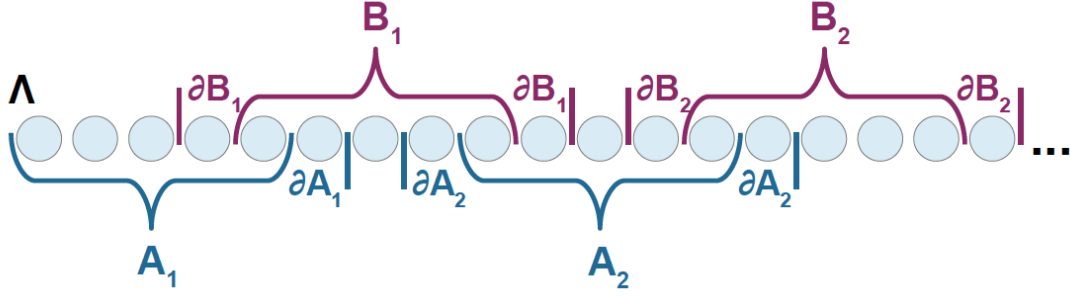


Figure 10.2: Splitting of Λ in fixed-sized subsets A_i and B_i , of which we just show the first four terms. We reduce for simplicity to the case $k = 2, l = 1$.

10.3 POSITIVITY OF THE LOG-SOBOLEV CONSTANT FOR THE HEAT-BATH DYNAMICS

In this section, we state and prove the main result of this chapter, namely a static sufficient condition on the Gibbs state of a k -local commuting Hamiltonian for the heat-bath dynamics in 1D to have a positive logarithmic Sobolev constant. For that, we first need to introduce two assumptions that need to be considered in order to prove the result, and which will be discussed in further detail in the next section, where we will identify them as the necessary clustering conditions on the Gibbs state for the positivity of the log-Sobolev constant to hold.

The first condition can be interpreted as an exponential decay of correlations in the Gibbs state of the commuting Hamiltonian. In Section 10.4.1 we will see that only a weaker assumption is necessary, although this form is preferable here for its close connections to its classical analogue [DPP02].

Assumption 10.3.1 — MIXING CONDITION, (Bar+19).

Let $\Lambda \subset \mathbb{Z}$ be a finite chain and let $C, D \subset \Lambda$ be the union of non-overlapping finite-sized segments of Λ . Let σ_Λ be the Gibbs state of a commuting Hamiltonian. The following inequality holds for certain positive constants K_1, K_2 independent on Λ, C, D :

$$\left\| \sigma_C^{-1/2} \otimes \sigma_D^{-1/2} \sigma_{CD} \sigma_C^{-1/2} \otimes \sigma_D^{-1/2} - \mathbb{1}_{CD} \right\|_\infty \leq K_1 e^{-K_2 d(C,D)},$$

where $d(C, D)$ is the distance between C and D , i.e., the minimum distance between two segments of C and D .

The second condition that needs to be assumed constitutes a strong form of quasi-factorization of the relative entropy.

Assumption 10.3.2 — STRONG QUASI-FACTORIZATION, (Bar+19).

Let $\Lambda \subset \mathbb{Z}$ be a finite chain and $X \subset \Lambda$. Let σ_Λ be the Gibbs state of a k -local commuting Hamiltonian. For every $\rho_\Lambda \in \mathcal{S}_\Lambda$, the following inequality holds

$$D_X(\rho_\Lambda \| \sigma_\Lambda) \leq f_X(\sigma_\Lambda) \sum_{x \in X} D_x(\rho_\Lambda \| \sigma_\Lambda), \quad (10.12)$$

where $1 \leq f_X(\sigma_\Lambda) < \infty$ depends only on σ_Λ and is independent of $|\Lambda|$.

This form of quasi-factorization is stronger than the ones presented in Chapter 7 and more similar to those of Chapter 8, since another conditional relative entropy appears in the LHS of the inequality, instead of a relative entropy as in the main results of the former chapter. Moreover, the error term depends only on the second state, as in usual quasi-factorization results, but only on its value in the regions where the relative entropies are being conditioned and their boundaries. In particular, it is independent of the size of the chain.

As in the case of Assumption 10.3.1, we will see in Subsection 10.4.2 that only a weaker condition is necessary for Theorem 10.3.3 to hold true, since this condition will only appear in the proof concerning sets X of small size.

Let us now state and prove the main result of this chapter, namely the positivity of the log-Sobolev constant for the heat-bath dynamics in 1D.

Theorem 10.3.3 — LOG-SOBOLEV CONSTANT FOR HEAT-BATH DYNAMICS IN 1D, (Bar+19).

Let $\Lambda \subset \mathbb{Z}$ be a finite chain. Let $\Phi : \Lambda \rightarrow \mathcal{A}_\Lambda$ be a k -local commuting potential, $H_\Lambda = \sum_{x \in \Lambda} \Phi(x)$ its corresponding Hamiltonian, and denote by σ_Λ its Gibbs state. Let \mathcal{L}_Λ^* be the generator of the heat-bath dynamics. Then, if Assumptions 10.3.1 and 10.3.2 hold, the log-Sobolev constant of \mathcal{L}_Λ^* is strictly positive and independent of $|\Lambda|$.

The proof of this result will be split into four parts. First, we need to define a splitting of the chain into two (not connected) subsets $A, B \subset \Lambda$, with a certain geometry so that:

1. They cover the whole chain.
2. Their intersection is large enough.
3. Each one of them is composed of smaller segments of fixed size, but large enough to contain two non-overlapping half-boundaries of two other segments, respectively.

More specifically, fix $l \in \mathbb{N}$ so that $K_1 e^{-K_2 l} < \frac{1}{2}$, for K_1 and K_2 the constants appearing in the mixing condition, and consider the splitting of Λ given in terms of A and B verifying the following conditions (see Figure 10.2):

1. $\Lambda = A \cup B$.
2. $A = \bigcup_{i=1}^n A_i$ and $B = \bigcup_{j=1}^n B_j$.
3. $|A_i \cap B_i| = |B_i \cap A_{i+1}| = l$ for every $i = 1, \dots, n-1$.
4. $|A_i| = |B_j| = 2(k+l) - 1$ for all $i, j = 1, \dots, n$, where k comes from the k -locality of the Hamiltonian.

Note that the total size of Λ is then $n(4k+2l-2) + l$ sites. Hence, fixing l and k as already mentioned, we can restrict our study here to lattices of size $n(4k+2l-2) + l$ for every $n \in \mathbb{N}$, as we will be interested in the scaling properties in the limit.

10.3.1 STEP 1: QUASI-FACTORIZATION OF THE ENTROPY INTO TWO REGIONS

In the first step, considering this decomposition of the chain, we show an upper bound for the relative entropy of two states on Λ (the second of them being the Gibbs state) in terms of the sum of two conditional relative entropies in A and B , respectively, and a multiplicative error term that measures how far the reduced state $\sigma_{A^c B^c}$ is from a tensor product between A^c and B^c , where $A^c := \Lambda \setminus A$ and $B^c := \Lambda \setminus B$.

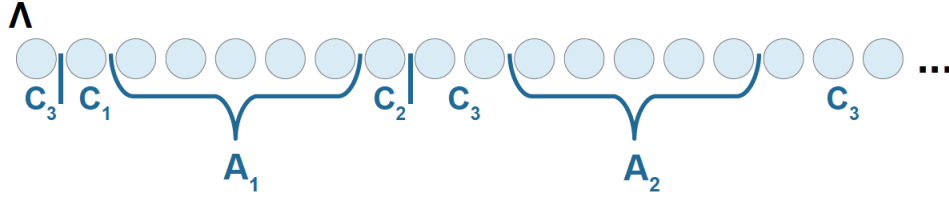


Figure 10.3: Splitting of A in fixed-sized subsets A_i so that their boundaries do not overlap. For simplicity we restrict to the case $k = 2, l = 1$.

Step 10.3.4 For the regions A and B defined above, and for any $\rho_\Lambda \in \mathcal{S}_\Lambda$, we have

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \frac{1}{1 - 2\|h(\sigma_{A^c B^c})\|_\infty} [D_A(\rho_\Lambda || \sigma_\Lambda) + D_B(\rho_\Lambda || \sigma_\Lambda)], \quad (10.13)$$

where

$$h(\sigma_{A^c B^c}) = \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} \sigma_{A^c B^c} \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} - \mathbb{1}_{A^c B^c}.$$

This step constitutes a reformulation of Theorem 7.3.1. See Chapter 7 for its proof.

10.3.2 STEP 2: QUASI-FACTORIZATION OF THE ENTROPY INTO MANY REGIONS

For the second step of the proof, we focus on one of the two components of Λ , e.g. A , and upper bound the conditional relative entropy of two states in the whole A in terms of the sum of the conditional relative entropies in its fixed-size small components. In this case, there is no multiplicative error term, due to the structure of quantum Markov chain of the Gibbs state between one component, its boundary, and the complement, and the fact that the boundaries of these components do not overlap.

Step 10.3.5 For $A = \bigcup_{i=1}^n A_i$ defined as above (see Figure 10.3), and for every $\rho_\Lambda \in \mathcal{S}_\Lambda$, the following holds:

$$D_A(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{i=1}^n D_{A_i}(\rho_\Lambda || \sigma_\Lambda). \quad (10.14)$$

Without loss of generality, we assume that $A = A_1 \cup A_2$ (the general result follows by induction in the number of subsets A_i). For these two subregions, as in the previous step, this result constitutes a reformulation of Theorem 8.2.1.

Combining expressions (10.13) and (10.14) from Steps 10.3.4 and 10.3.5, respectively, we get

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \frac{1}{1 - 2\|h(\sigma_{A^c B^c})\|_\infty} \sum_{i=1}^n [D_{A_i}(\rho_\Lambda || \sigma_\Lambda) + D_{B_i}(\rho_\Lambda || \sigma_\Lambda)], \quad (10.15)$$

This equation corresponds to the result of quasi-factorization of the relative entropy that constitutes the first part in the proof of the positive log-Sobolev constant (see Figure 10.4).

10.3.3 STEP 3: LOWER BOUND FOR THE LOG-SOBOLEV CONSTANT IN TERMS OF THE COND. ONE

In the third step of the proof, using the first two, we get a lower bound for the global log-Sobolev constant of the whole chain in terms of the conditional log-Sobolev constants on the

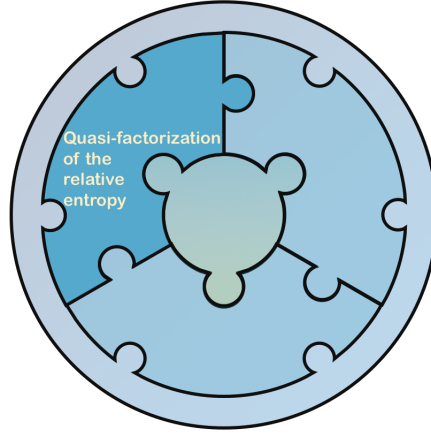


Figure 10.4: Piece for the quasi-factorization of the relative entropy. Note that we use here the image that we usually devote to strong results of quasi-factorization. Although in the left-hand side of the inequality there is no conditional relative entropy, the result is stronger than we would directly obtain from Step 1, and this justifies the use of this image.

aforementioned fixed-sized regions A_i and B_i . For that, we need to consider that Assumption 10.3.1 holds true.

Step 10.3.6 If Assumption 10.3.1 holds, we have:

$$\alpha(\mathcal{L}_\Lambda^*) \geq \tilde{K} \min_{i \in \{1, \dots, n\}} \{ \alpha_\Lambda(\mathcal{L}_{A_i}^*), \alpha_\Lambda(\mathcal{L}_{B_i}^*) \},$$

where $\tilde{K} = \frac{1 - 2K_1 e^{-K_2 l}}{2}$ and $\alpha_\Lambda(\mathcal{L}_{A_i}^*)$, resp. $\alpha_\Lambda(\mathcal{L}_{B_i}^*)$, denotes the conditional log-Sobolev constant of \mathcal{L}_Λ^* on A_i , resp. B_i , as introduced in Definition 10.1.3.

Proof. By Equation (10.15) and Assumption 10.3.1, we have

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \frac{1}{1 - 2K_1 e^{-K_2 l}} \sum_{i=1}^n [D_{A_i}(\rho_\Lambda || \sigma_\Lambda) + D_{B_i}(\rho_\Lambda || \sigma_\Lambda)]. \quad (10.16)$$

Now, by virtue of the definition of conditional log-Sobolev constants on each A_i and B_i , it is clear that

$$\begin{aligned} D(\rho_\Lambda || \sigma_\Lambda) &\leq \frac{1}{1 - 2K_1 e^{-K_2 l}} \sum_{i=1}^n \left[\frac{-\text{tr}[\mathcal{L}_{A_i}^*(\rho_\Lambda) (\log \rho_\Lambda - \log \sigma_\Lambda)]}{2\alpha_\Lambda(\mathcal{L}_{A_i}^*)} + \frac{-\text{tr}[\mathcal{L}_{B_i}^*(\rho_\Lambda) (\log \rho_\Lambda - \log \sigma_\Lambda)]}{2\alpha_\Lambda(\mathcal{L}_{B_i}^*)} \right] \\ &\leq \frac{1}{1 - 2K_1 e^{-K_2 l}} \frac{1}{2 \min_{i \in \{1, \dots, n\}} \{ \alpha_\Lambda(\mathcal{L}_{A_i}^*), \alpha_\Lambda(\mathcal{L}_{B_i}^*) \}} \sum_{i=1}^n [\text{EP}_{A_i}(\rho_\Lambda) + \text{EP}_{B_i}(\rho_\Lambda)]. \end{aligned}$$

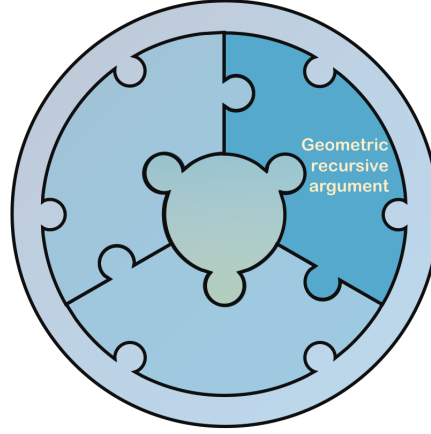


Figure 10.5: Piece for the geometric argument for the heat-bath dynamics.

Therefore,

$$\begin{aligned}
 2 \min_{i \in \{1, \dots, n\}} \{ \alpha_{\Lambda}(\mathcal{L}_{A_i}^*), \alpha_{\Lambda}(\mathcal{L}_{B_i}^*) \} D(\rho_{\Lambda} \| \sigma_{\Lambda}) \\
 \leq \frac{1}{1 - 2K_1 e^{-K_2 l}} \left[-\text{tr} \left[(\mathcal{L}_{\Lambda}^*(\rho_{\Lambda}) + \mathcal{L}_{A_n \cap B_n}^*(\rho_{\Lambda})) (\log \rho_{\Lambda} - \log \sigma_{\Lambda}) \right] \right. \\
 \left. + \sum_{i=1}^{n-1} (\mathcal{L}_{A_i \cap B_i}^*(\rho_{\Lambda}) + \mathcal{L}_{A_{i+1} \cap B_i}^*(\rho_{\Lambda})) (\log \rho_{\Lambda} - \log \sigma_{\Lambda}) \right] \\
 \leq \frac{2}{1 - 2K_1 e^{-K_2 l}} [-\text{tr}[\mathcal{L}_{\Lambda}^*(\rho_{\Lambda}) (\log \rho_{\Lambda} - \log \sigma_{\Lambda})]], \tag{10.17}
 \end{aligned}$$

where we have used the locality of the Lindbladian and the positivity of the entropy productions.

Finally, note that the last term of expression (10.17) is the entropy production of ρ_{Λ} . Hence, considering the quotient of this term over the relative entropy of the LHS, and taking infimum over $\rho_{\Lambda} \in \mathcal{S}_{\Lambda}$, we get

$$\alpha(\mathcal{L}_{\Lambda}^*) = \inf_{\rho_{\Lambda} \in \mathcal{S}_{\Lambda}} \frac{\text{EP}(\rho_{\Lambda})}{2D(\rho_{\Lambda} \| \sigma_{\Lambda})} \geq \tilde{K} \min_{i \in \{1, \dots, n\}} \{ \alpha_{\Lambda}(\mathcal{L}_{A_i}^*), \alpha_{\Lambda}(\mathcal{L}_{B_i}^*) \},$$

where $\tilde{K} := \frac{1 - 2K_1 e^{-K_2 l}}{2} > 0$. ■

This step represents the geometric recursive argument associated to Figure 10.5 for the strategy to obtain positivity for the log-Sobolev constant associated to the heat-bath generator. Not that, because of the geometry introduced in Figure 10.2, there is no need for a complex recursion in this step.

10.3.4 STEP 4: POSITIVE CONDITIONAL LOG-SOBOLEV CONSTANT FOR THE HEAT-BATH GEN.

Finally, in the last step of the proof, we show that the conditional log-Sobolev constants on every A_i and B_i are strictly positive and, additionally, independent of the size of Λ . For that, we need to suppose that Assumption 10.3.2 holds true. We also make use of some technical results from the previous section.

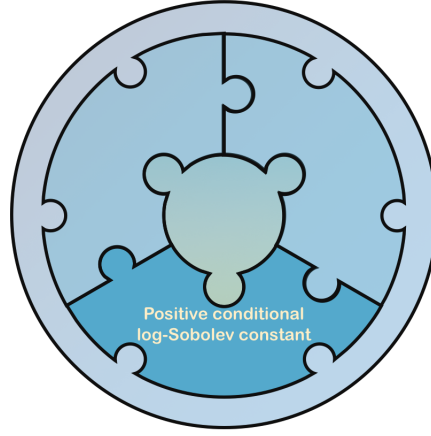


Figure 10.6: Piece for the positivity of the conditional log-Sobolev constant for the heat-bath dynamics.

Step 10.3.7 If Assumption 10.3.2 holds, for any A_i defined as above we have

$$\alpha_\Lambda(\mathcal{L}_{A_i}^*) \geq C_{A_i}(\sigma_\Lambda) > 0,$$

with $C_{A_i}(\sigma_\Lambda)$ independent of the size of Λ , and analogously for any B_i .

Proof. Consider $X \in \{A_i, B_i : 1 \leq i \leq n\}$. Let us first recall that the conditional log-Sobolev constant in X is given by

$$\begin{aligned} \alpha_\Lambda(\mathcal{L}_X^*) &= \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{\text{EP}_X(\rho_\Lambda)}{2D_X(\rho_\Lambda || \sigma_\Lambda)} \\ &= \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\sum_{x \in X} \text{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_X(\rho_\Lambda || \sigma_\Lambda)}. \end{aligned}$$

By virtue of Lemma 10.2.1, we have

$$\text{EP}_x(\rho_\Lambda) \geq D_x(\rho_\Lambda || \sigma_\Lambda)$$

for every $x \in X$, and, thus,

$$\alpha_\Lambda(\mathcal{L}_X^*) \geq \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{\sum_{x \in X} D_x(\rho_\Lambda || \sigma_\Lambda)}{2D_X(\rho_\Lambda || \sigma_\Lambda)}. \quad (10.18)$$

Note that the quotient in the RHS of (10.18) is well-defined, since we have seen in Corollary 10.2.6 that the kernel of $D_X(\rho_\Lambda || \sigma_\Lambda)$ coincides with the intersection of the kernels of $D_x(\rho_\Lambda || \sigma_\Lambda)$ for every $x \in X$. Furthermore, because of Assumption 10.3.2, we obtain the following lower bound for the conditional log-Sobolev constant

$$\alpha_\Lambda(\mathcal{L}_X^*) \geq \frac{1}{2f_X(\sigma_\Lambda)}, \quad (10.19)$$

which is strictly positive, only depends on σ_Λ and does depend on the size of Λ . ■

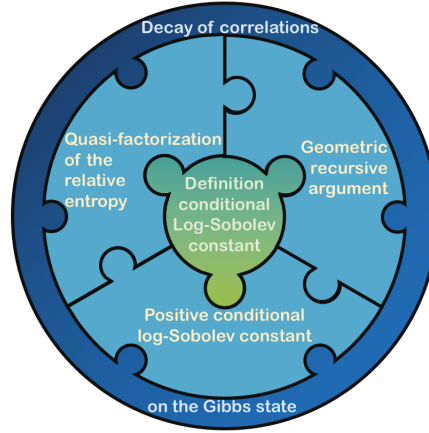


Figure 10.7: Complete puzzle for the positivity of the log-Sobolev constant for the heat-bath dynamics.

This last step of the proof represents the piece of positivity of the conditional log-Sobolev constant (see Figure 10.6).

Finally, putting together Steps 10.3.4 and 10.3.5 (piece of quasi-factorization), 10.3.6 (piece of geometric recursive argument) and 10.3.7 (piece of positive conditional log-Sobolev constant), along with the definition of conditional log-Sobolev constant and Assumptions 10.3.1 and 10.3.2 (piece of decay of correlations on the Gibbs state), we conclude the proof of Theorem 10.3.3 (see Figure 10.7).

10.4 MIXING CONDITION AND STRONG QUASI-FACTORIZATION

In the next two subsections, we will discuss the two conditions related to the decay of correlations on the Gibbs state that we have assumed for the log-Sobolev constant of the heat-bath dynamics in 1D to be positive (see Figure 10.8).

10.4.1 MIXING CONDITION

In this subsection, we will elaborate on the mixing condition introduced in Assumption 10.3.1 and provide sufficient conditions for it to hold. Consider $\Lambda \subset \mathbb{Z}$ a finite chain and $A, B \subset \Lambda$ as in the splitting of Λ in the proof of Theorem 10.3.3 (see Figure 10.2). Denote $C := B^c$ and $D := A^c$, so that they can be expressed as the union of disjoint segments, $C = \bigcup_{i=1}^n C_i$ and

$D = \bigcup_{j=1}^n D_j$, respectively. For every $i = 1, \dots, n-1$, denote by E_i , respectively F_i , the connected set that separate C_i from D_i , respectively D_i from C_{i+1} (see Figure 10.9). Note that, because of the construction of A and B described in the previous section, every E_i and F_i is composed of, at least, $2k-1$ sites.

Let σ_Λ be the Gibbs state of a k -local commuting Hamiltonian. Then, with this construction, Assumption 10.3.1 can be read as the existence of positive constants K_1, K_2 independent of Λ for which the following holds:

$$\left\| \sigma_C^{-1/2} \otimes \sigma_D^{-1/2} \sigma_{CD} \sigma_C^{-1/2} \otimes \sigma_D^{-1/2} - \mathbb{1}_{CD} \right\|_\infty \leq K_1 e^{-K_2 l}, \quad (\text{A1})$$

where $l = d(C, D)$.

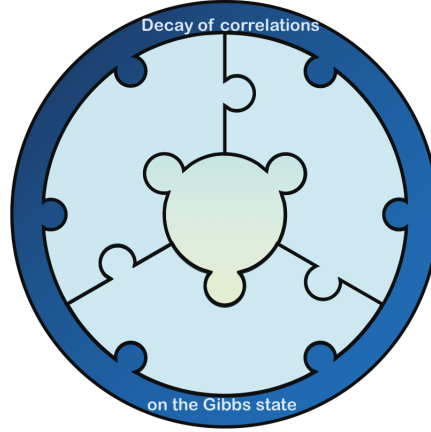


Figure 10.8: Conditions of decay of correlations on the Gibbs state.

This exponential decay of correlations on the Gibbs state is similar to certain forms of decay of correlations of states that frequently appear in the literature of both classical and quantum spin systems. In the latter, this is closely related, for instance, to the concept of *LTQO* (*Local Topological Quantum Order*) [MP13], or the *local indistinguishability* that was introduced in [KB16].

The main difference with the (strong) *mixing condition* of the classical case [DPP02] lies in the fact that they considered a decay of correlations with the distance between two connected regions (in particular, rectangles), whereas in our case we have a finite union of regions of that kind. The fact that the regions are connected is essential for some other properties that can be derived from the *Dobrushin condition* ([DS87] [Mar99] [DS85, Condition III.d]), but not to derive the decay of correlations on the Gibbs measure with the distance between the regions considered, as was shown in [Ces01]. Hence, we have the following proposition.

Proposition 10.4.1 Let σ_Λ be the Gibbs state of a k -local commuting Hamiltonian and assume that σ_Λ is a classical. Then, Condition (A1) holds.

Nevertheless, the mixing condition that we need to assume for the proof of Theorem 10.3.1 to hold is actually a bit weaker. Indeed, the only necessary thing is that we can bound the LHS of (A1) by something that is strictly smaller than $1/2$, i.e.,

$$\left\| \sigma_C^{-1/2} \otimes \sigma_D^{-1/2} \sigma_{CD} \sigma_C^{-1/2} \otimes \sigma_D^{-1/2} - \mathbb{1}_{CD} \right\|_\infty < \frac{1}{2}, \quad (\text{A1-weaker})$$

It is clear that (A1) implies (A1-weaker), as one can always choose l big enough. This new condition is a bit more approachable and we will show below that states with a defect at site i so that the interaction is bigger there, but interactions decay away from that site, satisfy this condition.

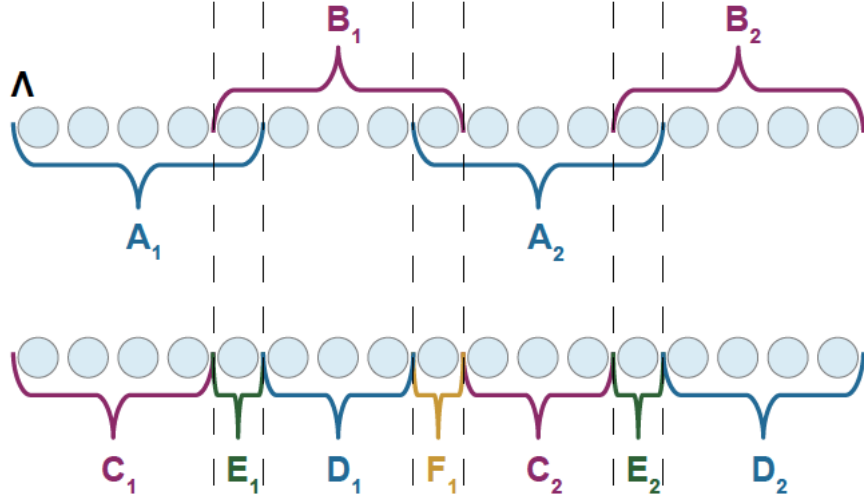


Figure 10.9: Notation introduced in the splitting of Λ into size-fixed A_i and B_i for the discussion in Assumption 10.3.1. For simplicity we restrict to the case $k = 2, l = 1$.

Proposition 10.4.2 — (Bar+19).

Let Λ be a finite chain and consider a splitting on it as the one of Figure 10.9. If we assume the following condition:

$$\prod_{i=1}^n \gamma_i^2 > \frac{2}{3} \prod_{i=1}^n \delta_i^2 > \frac{1}{3},$$

where we are writing

- $\gamma_i := \gamma_{CE}^{(i)} \gamma_{ED}^{(i)} \gamma_{DF}^{(i)} \gamma_{FC}^{(i)}$, for $i = 1, \dots, n-1$,
- $\delta_i := \delta_{CE}^{(i)} \delta_{ED}^{(i)} \delta_{DF}^{(i)} \delta_{FC}^{(i)}$, for $i = 1, \dots, n-1$,
- $\gamma_n := \gamma_{CE}^{(n)} \gamma_{ED}^{(n)}$,
- $\delta_n := \delta_{CE}^{(n)} \delta_{ED}^{(n)}$,

and for which each $\gamma_{GH}^{(i)}$, resp. $\delta_{GH}^{(i)}$, is the minimum, resp. maximum, eigenvalue of $\sigma_{(\partial G_i) \cap H_i}$, then (A1-weaker) holds.

Proof. First, note that condition (A1-weaker) is equivalent to the following

$$\frac{1}{2} \sigma_C \otimes \sigma_D < \sigma_{CD} < \frac{3}{2} \sigma_C \otimes \sigma_D. \quad (10.20)$$

Now, the state σ_Λ on the full chain can be decomposed into the following product of commuting terms (see Figure 10.10):

$$Z \sigma_\Lambda := \left(\prod_{i=1}^{n-1} \chi_i \right) \tilde{\sigma}_{C_n}^\circ \tilde{\sigma}_{(\partial C_n) \cap E_n} \tilde{\sigma}_{E_n}^\circ \tilde{\sigma}_{(\partial D_n) \cap E_n} \tilde{\sigma}_{D_n}^\circ, \quad (10.21)$$

with

$$\chi_i := \tilde{\sigma}_{C_i}^\circ \tilde{\sigma}_{(\partial C_i) \cap E_i} \tilde{\sigma}_{E_i}^\circ \tilde{\sigma}_{(\partial D_i) \cap E_i} \tilde{\sigma}_{D_i}^\circ \tilde{\sigma}_{(\partial D_i) \cap F_i} \tilde{\sigma}_{F_i}^\circ \tilde{\sigma}_{(\partial C_{i+1}) \cap F_i}, \quad (10.22)$$

and where Z is the normalization factor and \mathring{G} denotes the interior of G , that is the set of sites in G whose corresponding interaction is fully supported in G . We use the notation $\tilde{\sigma}_G$ to

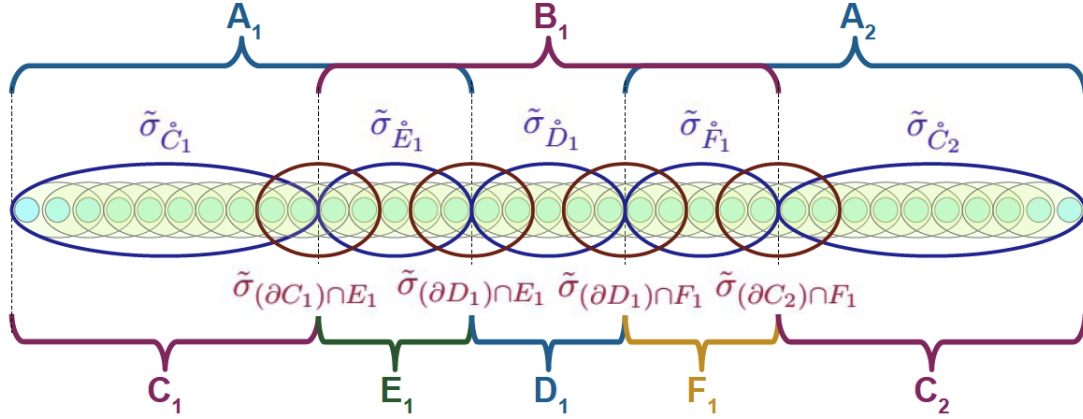


Figure 10.10: Decomposition of σ_Λ into the product of commuting terms for $k = 3$ and $l = 5$, assuming that Λ is decomposed only into A_1, B_1 and A_2 for simplification.

remark that this term does not coincide in general with $\text{tr}_{G^c}[\sigma_\Lambda]$. We will bound the boundary terms as follows: For any consecutive $G_j, H_i \in \{C_j, D_i, E_i, F_i\}$ so that $H_i = E_i$ or F_i (and thus $G_j = C_i, C_{i+1}$ or D_i), we have

$$\gamma_{GH}^{(i)} \mathbb{1}_{(\partial G_j) \cap H_i} \leq \tilde{\sigma}_{(\partial G_j) \cap H_i} \leq \delta_{GH}^{(i)} \mathbb{1}_{(\partial G_j) \cap H_i}. \quad (10.23)$$

Note that, in a slight abuse of notation, we are denoting by $\gamma_{FC}^{(i)}$ and $\delta_{FC}^{(i)}$ the coefficients corresponding to the term $\tilde{\sigma}_{F_i \cap (\partial C_{i+1})}$. Then, since $(\partial G_j) \cap H_i$ consists of $2(k-1)$ sites, half of which belong to G_j and the other half to H_i , we can write

$$\gamma_{GH}^{(i)} \tilde{\sigma}_{G_i} \otimes \tilde{\sigma}_{H_i} \leq \tilde{\sigma}_{G_i} \tilde{\sigma}_{(\partial G_i) \cap H_i} \tilde{\sigma}_{H_i} \leq \delta_{GH}^{(i)} \tilde{\sigma}_{G_i} \otimes \tilde{\sigma}_{H_i}$$

and thus replacing (10.23) in (10.21) after tracing out E and F , it is easy to show that

$$\begin{aligned} & \left(\prod_{i=1}^{n-1} \gamma_i \tilde{\sigma}_{C_i D_i} \text{tr}(\tilde{\sigma}_{E_i F_i}) \right) \gamma_{CE}^{(n)} \gamma_{ED}^{(n)} \tilde{\sigma}_{C_n D_n} \text{tr}(\tilde{\sigma}_{E_n}) \\ & \leq Z \sigma_{CD} \\ & \leq \left(\prod_{i=1}^{n-1} \delta_i \tilde{\sigma}_{C_i D_i} \text{tr}(\tilde{\sigma}_{E_i F_i}) \right) \delta_{CE}^{(n)} \delta_{ED}^{(n)} \tilde{\sigma}_{C_n D_n} \text{tr}(\tilde{\sigma}_{E_n}), \end{aligned}$$

where $\gamma_i := \gamma_{CE}^{(i)} \gamma_{ED}^{(i)} \gamma_{DF}^{(i)} \gamma_{FC}^{(i)}$ and $\delta_i := \delta_{CE}^{(i)} \delta_{ED}^{(i)} \delta_{DF}^{(i)} \delta_{FC}^{(i)}$, and d is the dimension of the local Hilbert space associated to each site.

On the other hand, if we proceed analogously to get a bound for $\sigma_C \otimes \sigma_D$ to compare it with σ_{CD} , we obtain

$$\begin{aligned} & \left(\prod_{i=1}^{n-1} \gamma_i^2 \tilde{\sigma}_{C_i} \tilde{\sigma}_{D_i} \text{tr}(\tilde{\sigma}_{E_i F_i})^2 \right) \left(\gamma_{CE}^{(n)} \gamma_{ED}^{(n)} \right)^2 \tilde{\sigma}_{C_n} \tilde{\sigma}_{D_n} \text{tr}(\tilde{\sigma}_{CD}) \\ & \leq Z^2 \sigma_C \otimes \sigma_D \\ & \leq \left(\prod_{i=1}^{n-1} \delta_i^2 \tilde{\sigma}_{C_i} \tilde{\sigma}_{D_i} \text{tr}(\tilde{\sigma}_{E_i F_i})^2 \right) \left(\delta_{CE}^{(n)} \delta_{ED}^{(n)} \right)^2 \tilde{\sigma}_{C_n} \tilde{\sigma}_{D_n} \text{tr}(\tilde{\sigma}_{CD}). \end{aligned}$$

Therefore, a sufficient condition for (A1-weaker) is that

$$\frac{1}{2} \operatorname{tr}(\sigma_{\hat{C}\hat{D}}) \prod_{i=1}^{n-1} \frac{\gamma_i \operatorname{tr}(\sigma_{\hat{E}_i \hat{F}_i})}{\delta_i^2} \frac{\gamma_n}{\delta_n^2} < Z < \frac{3}{2} \operatorname{tr}(\sigma_{\hat{C}\hat{D}}) \prod_{i=1}^{n-1} \frac{\gamma_i^2 \operatorname{tr}(\sigma_{\hat{E}_i \hat{F}_i})}{\delta_i} \frac{\gamma_n^2}{\delta_n}. \quad (10.24)$$

with $\gamma_n := \gamma_{CE}^{(n)} \gamma_{ED}^{(n)}$ and $\delta_n := \delta_{CE}^{(n)} \delta_{ED}^{(n)}$. Note that, when $\beta \rightarrow 0$, $Z \rightarrow d^{|\Lambda|}$, where the number $|\Lambda|$ of sites is equal to $|\hat{E}| + |\hat{F}| + 8(k-1)(n-1) + |\hat{C}| + |\hat{D}|$. Moreover, $\delta_i = \gamma_i = 1$ in the limit. Therefore (10.24) holds trivially, since it reduces to $\frac{1}{2} < 1 < \frac{3}{2}$. It is reasonable then to think that, close to infinite temperature, (10.24) holds.

Indeed, let us assume the following inequality between the γ_i and δ_i ,

$$\prod_{i=1}^n \gamma_i^2 > \frac{2}{3} \prod_{i=1}^n \delta_i^2 > \frac{1}{3}. \quad (10.25)$$

To conclude the proof that Equation (10.25) implies Equation (A1-weaker), it is enough to bound Z , the normalization factor, in the same way that we have bounded σ_{CD} and $\sigma_C \otimes \sigma_D$. Introducing those bounds in the inequalities appearing in (10.24), it is easy to see that this expression reduces to (10.25). ■

10.4.2 STRONG QUASI-FACTORIZATION

In this subsection, we will discuss Assumption 10.3.2, which can be seen as a strong quasi-factorization of the relative entropy, and provide some sufficient conditions on σ_Λ for it.

Given Λ a finite chain and A a subset of Λ , if we denote by σ_Λ the Gibbs state of a k -local commuting Hamiltonian, Assumption 10.3.2 reads as:

$$D_A(\rho_\Lambda || \sigma_\Lambda) \leq f_A(\sigma_\Lambda) \sum_{x \in A} D_x(\rho_\Lambda || \sigma_\Lambda) \quad \forall \rho_\Lambda \in \mathcal{S}_\Lambda, \quad (10.26)$$

where $1 \leq f_A(\sigma_\Lambda) < \infty$ depends only on σ_Λ and is independent of $|\Lambda|$.

Let us first recall that A has a fixed size of $2(k+l)-1$ sites, so $|A\partial| = 2(2k+l-1)-1$, and is, in particular, fixed. Moreover, if we separate one site from the rest in each step, i.e., for every $2 \leq m \leq |A|$, if we consider the only connected $B^{(m)} \in A$ of size m that contains the first site of A , and we split $B^{(m)}$ into two connected regions $B_1^{(m)}$ and $B_2^{(m)}$ so that $|B_1^{(m)}| = 1$, it is clear that the following inequality

$$D_{B^{(m)}}(\rho_\Lambda || \sigma_\Lambda) \leq f_{B^{(m)}}(\sigma_\Lambda) \left[D_{B_1^{(m)}}(\rho_\Lambda || \sigma_\Lambda) + D_{B_2^{(m)}}(\rho_\Lambda || \sigma_\Lambda) \right] \quad \forall \rho_\Lambda \in \mathcal{S}_\Lambda, \quad (10.27)$$

implies inequality (10.26) by induction, taking

$$f_A(\sigma_\Lambda) := \sup_{2 \leq m \leq |A|} f_{B^{(m)}}(\sigma_\Lambda).$$

Therefore, we can pose the following natural question.

Question 10.4.3 Given two adjacent subsets $A, B \subset \Lambda$, can we impose any condition on the Gibbs state σ_Λ so that there exist a bounded $f_{AB}(\sigma_\Lambda)$ only depending on σ_Λ and independent of the size of Λ such that the following inequality holds for every $\rho_\Lambda \in \mathcal{S}_\Lambda$:

$$D_{AB}(\rho_\Lambda || \sigma_\Lambda) \leq f_{AB}(\sigma_\Lambda) (D_A(\rho_\Lambda || \sigma_\Lambda) + D_B(\rho_\Lambda || \sigma_\Lambda)) ? \quad (10.28)$$

Remark that we only need to answer this question for $|A|, |B| < 2(k+l)$. Although we cannot give a general answer to this problem, we can provide some motivation for situations in which it might hold. For that, we prove before the following lemma, which shows that a conditional relative entropy in a certain region can be upper bounded by a quantity depending only on the reduced states in that region independently of the cardinality of the whole lattice.

Lemma 10.4.4 — (Bar+19).

Let $A \subset \Lambda$. For any $\rho_\Lambda \in \mathcal{S}_\Lambda$,

$$D_A(\rho_\Lambda \| \sigma_\Lambda) \leq D_A(\rho_\Lambda \| \sigma_A \otimes \sigma_{A^c}) + D(\rho_{A\partial} \| \sigma_{A\partial}).$$

Proof. A simple use of the definition of the conditional relative entropy leads to the following identity:

$$\begin{aligned} D_A(\rho_\Lambda \| \sigma_\Lambda) - D_A(\rho_\Lambda \| \sigma_A \otimes \sigma_{A^c}) &= D(\rho_\Lambda \| \sigma_\Lambda) - D(\rho_\Lambda \| \sigma_A \otimes \sigma_{A^c}) \\ &= \text{tr}[\rho_\Lambda (-\log \sigma_\Lambda + \log \sigma_A \otimes \sigma_{A^c})]. \end{aligned} \quad (10.29)$$

By the quantum Markov chain property of the state σ_Λ between $A \leftrightarrow \partial A \leftrightarrow (A\partial)^c$ and by Proposition 4.7.5, we have

$$\log \sigma_\Lambda = \log \sigma_{A^c} + \log \sigma_{A\partial} - \log \sigma_{\partial A}.$$

Plugging this in Equation (10.29) we arrive at:

$$\begin{aligned} D_A(\rho_\Lambda \| \sigma_\Lambda) - D_A(\rho_\Lambda \| \sigma_A \otimes \sigma_{A^c}) &= \text{tr}[\rho_\Lambda (-\log \sigma_{A\partial} + \log \sigma_A \otimes \sigma_{\partial A})] \\ &= D(\rho_{A\partial} \| \sigma_{A\partial}) - D(\rho_{A\partial} \| \sigma_A \otimes \sigma_{\partial A}) \\ &\leq D(\rho_{A\partial} \| \sigma_{A\partial}). \end{aligned}$$

■

Note that, for ρ a classical density matrix, inequality (10.28) holds true for any Gibbs state of a classical k -local commuting Hamiltonian in 1D, and under some further assumptions it also does in more general dimensions, since (10.28) coincides in the classical setting with a usual result of quasi-factorization of the entropy, due to the DLR conditions. More specifically, this inequality holds classically whenever the Dobrushin-Shlosman complete analyticity condition holds. Moreover, in that setting one can see that $f_{AB}(\sigma_\Lambda)$ actually depends only on $\sigma_{(AB)\partial}$.

It is then reasonable to believe that this inequality might also hold true for Gibbs states of quantum k -local commuting Hamiltonians in 1D, although f_{AB} could possibly depend on σ on the whole lattice Λ (without depending on its size). The intuition behind this is that σ_Λ is also a quantum Markov chain, and Lemma 10.4.4 shows that the conditional relative entropy in a certain region can be approximated by its analogue for σ_Λ a tensor product obtaining an additive error term that can be bounded by something that only depends on the region and its boundary.

However, if we define

$$f_{AB}(\sigma_\Lambda) := \sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{D_{AB}(\rho_\Lambda \| \sigma_\Lambda)}{D_A(\rho_\Lambda \| \sigma_\Lambda) + D_B(\rho_\Lambda \| \sigma_\Lambda)}$$

we lack a proof that, in general, it satisfies the necessary conditions for (10.28) to hold. The study of examples of Hamiltonians whose Gibbs state satisfies the aforementioned inequality is left for future work.

Nevertheless, let us recall here some situations for which we already know that inequality (10.28) holds. First, if σ_Λ is a tensor product, this inequality holds with $f = 1$ (see Chapter 9), as

a consequence of strong subadditivity. Moreover, for a more general σ_Λ , if A and B are separated enough, we have seen in Step 5.2.5 that it also holds with $f = 1$, due to the structure of quantum Markov chain of σ_Λ . Since in (10.28) we are assuming that A and B are adjacent, we cannot use this property to “separate” A from B , i.e. write σ_Λ as a direct sum of tensor products that separate A from B , and thus the proof of Step 5.2.5 cannot be used here.

10.5 EXTENSION TO A LARGER DIMENSION

Take $\Lambda \subset \mathbb{Z}^d$ a finite d -dimensional lattice, for $d > 1$, and consider the problem of proving positivity of the log-Sobolev constant for the heat-bath dynamics associated to Λ following the same approach followed in this chapter.

First, we would need to cover a n -dimensional lattice with small rectangles overlapping pairwise in an analogous way to the construction described here for dimension 1. It is easy to realize that, even in dimension 2, one would need at least three systems to classify the small rectangles so that two belonging to the same class would not overlap. In general dimension, a short calculation by induction shows that if the number of systems required for the analogous construction to hold in dimension d is denoted by a_d , then it can be obtained through the following recursive formula:

$$a_d = 2a_{d-1} - 1 \quad \text{for all } d \geq 2,$$

in which $a_d \geq 3$ for every $d \geq 2$. Thus, for our strategy to hold in dimension, at least, 2, we would need a result of quasi-factorization that provides an upper bound for the relative entropy of two states in terms of the sum of three conditional relative entropies, instead of two, and a multiplicative error term. Since we are lacking a result of this kind so far, this approach constitutes an open problem.

Another possible approach to follow in dimension d for the heat-bath dynamics would be the analogue to the one used for the Davies dynamics in Chapter 11. In this case, the geometric splitting and the recursive argument employed hold for any dimension. However, this approach has the counterpart that it needs a result of strong quasi-factorization of the relative entropy to be carried out. If we managed to prove a result of this kind for either some conditional relative entropies or some conditional relative entropies by expectations, we would follow the steps of Chapter 11 to reduce the positivity of the global log-Sobolev constant to the conditional one, and would conclude by assuming analogous conditions of clustering of correlations to the ones presented here in 1D.



11. DAVIES DYNAMICS

In this chapter, we study the positivity of the log-Sobolev constant associated to the Davies dynamics. More specifically, we address the problem of finding conditions on the algebra of invariant states of a quantum dissipative evolution associated to the Davies dynamics conditioned to a sublattice of a greater lattice which imply positivity on the log-Sobolev constant associated to the global dynamics. In the next sections, we show that two different conditions of clustering of correlations lead to the desired result.

The main difference with respect to the situation addressed in Chapter 10 for the heat-bath dynamics lies in the fact that, under both conditions of clustering of correlations mentioned above, we manage to obtain results of strong quasi-factorization of the relative entropy, whereas in the latter case we only obtained results of weak quasi-factorization of the relative entropy. This translates in a different geometric recursive argument to reduce from the global log-Sobolev constant to the conditional one, since in the Davies case we can employ a standard recursive procedure, similar to those appearing on [DPP02], [Ces01] for classical spin systems, and [KB16] for quantum ones, as opposed to the heat-bath case, in which we had to devise a recursive procedure based on a more elaborated initial geometric splitting that had the counterpart of not allowing the result to hold in dimension greater than 1.

We have divided the proof of the positivity of the log-Sobolev constant for the Davies dynamics into two different chapters, namely Chapter 8 and the current one, due to the independent interest of the results presented in the former chapter (apart from their use to prove positivity of the log-Sobolev constant for the Davies dynamics) and to ease the comprehension of the procedure followed to obtain the main result. In the former chapter, two parts of the strategy presented in Section 1.2 were already addressed, namely the conditions of clustering of correlations that need to be assumed and the results of quasi-factorization that follow from them. In the first section of this chapter, we review Davies dynamics and present some properties that will be of use in further sections. In Section 11.2, we reduce the problem of positivity of the global log-Sobolev constant to the conditional one, using a geometric recursive argument that, as mentioned above, holds for any finite dimension. Subsequently, we discuss the end of the proof of positivity of the log-Sobolev constant based on a conjecture. Finally, we conclude in Section 11.4 by showing an example of a physical system satisfying one of the conditions of clustering of correlations mentioned above.

This is an amazing view of the Niagara Falls, close to Waterloo (Canada), where I attended the workshop *Quantum Innovators in math and computer science* in October 2019.

11.1 DAVIES GENERATORS

Davies generators model the dynamics resulting from the weak coupling limit of a system in contact with a large heat-bath. Let us recall how to describe their structure: Given a finite lattice $\Lambda \subset \mathbb{Z}^d$, define the tensor product Hilbert space $\mathcal{H} := \mathcal{H}_\Lambda \equiv \bigotimes_{k \in \Lambda} \mathcal{H}_k$, where for each $k \in \Lambda$, $\mathcal{H}_k \simeq \mathbb{C}^\ell$, $\ell \in \mathbb{N}$. Then, let $\Phi : \Lambda \rightarrow \mathcal{A}_\Lambda$ be an r -local potential, i.e. for any $j \in \Lambda$, $\Phi(j)$ is self-adjoint and supported on a ball of radius r around site j . We assume further that $\|\Phi(j)\| \leq K$ for some constant $K < \infty$. Recall that the potential Φ is said to be a *commuting potential* if for any $i, j \in \Lambda$, $[\Phi(i), \Phi(j)] = 0$.

Given such a commuting potential, the Hamiltonian on a subregion $A \subseteq \Lambda$ is defined as

$$H_A := \sum_{j \in A} \Phi(j). \quad (11.1)$$

Hence, the corresponding Gibbs state corresponding to the region A and at inverse temperature β is defined as

$$\sigma_A^\beta := \frac{e^{-\beta H_A}}{\text{tr}(e^{-\beta H_A})}. \quad (11.2)$$

Note that this is in general not equal to the state $\text{tr}_B(\sigma_A^\beta)$. Indeed, if we define by

$$\partial A := \{j \in \Lambda \setminus A \mid \text{supp}(\Phi(j)) \cap A \neq \emptyset\},$$

the outer boundary of A and write $A\partial := A \cup \partial A$, it is clear that both H_A and σ_A^β are supported on $A\partial$.

Consider now the Hamiltonian $H_\Lambda := H_\Lambda^\Sigma$ of the system on the lattice Λ , the Hamiltonian H^{HB} of the heat-bath, as well as a set of system-bath interactions $\{S_{\alpha,k} \otimes B_{\alpha,k}\}$, where α labels all the operators $S_{\alpha,k}$ and $B_{\alpha,k}$ associated to the site $k \in \Lambda$, so that the Hamiltonian of the universe composed of the system and its heat-bath is given by

$$H = H_\Lambda + H^{\text{HB}} + \sum_{\alpha, k \in \Lambda} S_{\alpha,k} \otimes B_{\alpha,k}. \quad (11.3)$$

Here, we assume that the operators $S_{\alpha,k}$ form an orthonormal basis of self-adjoint operators in $\mathcal{A}(\mathcal{H})$ with respect to the Hilbert-Schmidt inner product (think of the qubit Pauli matrices). The operator $C_2 = \sum_\alpha S_{\alpha,k}^* S_{\alpha,k}$ is a Casimir operator of the Lie algebra $\mathfrak{su}(\ell)$ in the defining irreducible representation, and is hence proportional to the identity: $C_2 = \ell \mathbb{1}_{\mathcal{H}_k}$. In fact, the following holds:

Lemma 11.1.1 Let $\{S_\alpha\}_\alpha$ be an orthonormal basis of $\mathcal{A}(\mathbb{C}^\ell)$ endowed with the Hilbert-Schmidt scalar product. Then, for any $X \in \mathcal{B}(\mathbb{C}^\ell)$,

$$\sum_\alpha S_\alpha^* X S_\alpha = \text{tr}(X). \quad (11.4)$$

Proof. We use the fact that the operators $\{S_\alpha\}_\alpha$ form an orthonormal basis of operators in $\mathcal{A}(\mathbb{C}^\ell)$, so that there exists a unitary transformation from that basis to the orthonormal basis

$$\{V_k\}_k \equiv \left\{ \frac{1}{2}(E_{ij} + E_{ji}), \frac{i}{2}(E_{ij} - E_{ji}) \right\}_{i,j},$$

where $E_{ij} = |i\rangle\langle j| \in \mathcal{B}(\mathbb{C}^\ell)$ for every $i, j \in \{1, \dots, \ell\}$. More precisely, there exist scalars $\{\lambda_{\alpha k}\}$ such that $\sum_\alpha \bar{\lambda}_{\alpha k} \lambda_{\alpha k'} = \delta_{kk'}$ for any k, k' , and

$$S_\alpha = \sum_k \lambda_{\alpha k} V_k.$$

Then,

$$\sum_\alpha S_\alpha^* X S_\alpha = \sum_{\alpha, k_1, k_2} \bar{\lambda}_{\alpha k_1} \lambda_{\alpha k_2} V_{k_1}^* X V_{k_2} = \sum_k V_k^* X V_k. \quad (11.5)$$

Moreover, one can easily verify that the above right-hand side is equal to $\text{tr}(X)$. \blacksquare

If we further assume that the bath is in a Gibbs state, by a standard argument (e.g. weak coupling limit, see [SL78]), the evolution on the system can be approximated by a quantum Markov semigroup whose generator is of the following form:

$$\mathcal{L}_\Lambda^\beta(X) = i[H_\Lambda, X] + \sum_{k \in \Lambda} \mathcal{L}_k^\beta(X), \quad (11.6)$$

where

$$\mathcal{L}_k^\beta(X) = \sum_{\omega, \alpha} \chi_{\alpha, k}^\beta(\omega) \left(S_{\alpha, k}^*(\omega) X S_{\alpha, k}(\omega) - \frac{1}{2} \{S_{\alpha, k}^*(\omega) S_{\alpha, k}(\omega), X\} \right). \quad (11.7)$$

Then, the Fourier coefficients of the two-point correlation functions of the environment $\chi_{\alpha, k}^\beta$ satisfy the following KMS condition

$$\chi_{\alpha, k}^\beta(-\omega) = e^{-\beta\omega} \chi_{\alpha, k}^\beta(\omega).$$

Moreover, the operators $S_{\alpha, k}(\omega)$ are the Fourier coefficients of the system couplings $S_{\alpha, k}$, which means that they satisfy the following equation for any $t \in \mathbb{R}$:

$$e^{-itH_\Lambda} S_{\alpha, k} e^{itH_\Lambda} = \sum_\omega e^{it\omega} S_{\alpha, k}(\omega),$$

where the sum is over a finite number of frequencies, independent of the lattice size and for a commuting, local Hamiltonian. This implies in particular the following useful relation:

$$\Delta_\sigma(S_{\alpha, k}(\omega)) = e^{\beta\omega} S_{\alpha, k}(\omega).$$

The above identity yields the fact that the operators $S_{\alpha, k}(\omega)$ form a basis of eigenvectors of Δ_σ .

Analogously to the definition of the global generator for the Davies dynamics shown in (11.6), we can define the generator \mathcal{L}_A^β by restricting the sum above to the sublattice A :

$$\mathcal{L}_A^\beta(X) = i[H_A, X] + \sum_{k \in A} \mathcal{L}_k^\beta(X). \quad (11.8)$$

Note that \mathcal{L}_A^β acts non-trivially on $A\partial$. Then, for any region $A \subset \Lambda$, we define the *conditional expectation* onto the algebra \mathcal{N}_A of fixed points of \mathcal{L}_A^β with respect to the Gibbs state $\sigma = \sigma_\Lambda^\beta$ as follows [KB16]

$$\mathcal{E}_A^\beta(X) := \mathcal{E}(X | \mathcal{N}_A) = \lim_{t \rightarrow \infty} e^{t\mathcal{L}_A^\beta}(X). \quad (11.9)$$

It was shown in Lemma 11 of [KB16] that the generator of the Davies semigroups corresponding to a local commuting potential is *frustration-free*. This means that the state σ is invariant with respect to any \mathcal{L}_A^β , $A \subseteq \Lambda$. Therefore, the conditional expectations \mathcal{E}_A^β are all defined with respect to σ .

Finally, let us recall from Definition 8.3.1 that the conditional covariance in this case is denoted as follows, for any state $\rho_\Lambda \in \mathcal{S}_\Lambda$,

$$\text{Cov}_{\mathcal{N}_A, \rho_\Lambda}(X, Y) := \text{Cov}_{A, \rho_\Lambda}(X, Y) = \langle X - \mathcal{E}_A^\beta(X), Y - \mathcal{E}_A^\beta(Y) \rangle_{\rho_\Lambda}. \quad (11.10)$$

This definition is essential for the proof of positivity of the spectral gap of the Davies dynamics in [KB16] and will constitute the base for the conditions of clustering of correlation that we will need to assume for the analogous problem for the log-Sobolev constant (see Section 8.3).

Now, given a finite lattice Λ and $A \subset \Lambda$, take $\mathcal{N} = \mathcal{E}_A(\mathcal{B}_\Lambda)$ to be the so-called *decoherence-free subalgebra* of the non-primitive QMS $(\mathcal{P}_t^A = e^{t\mathcal{L}_A^\beta})_{t \geq 0}$, which is given by

$$\mathcal{N}(\mathcal{P}^A) := \{X \in \mathcal{B}_\Lambda \mid \mathcal{P}_t^A(X^*X) = \mathcal{P}_t^A(X)^* \mathcal{P}_t^A(X) \text{ and } \mathcal{P}_t^A(XX^*) = \mathcal{P}_t^A(X) \mathcal{P}_t^A(X)^* \forall t \geq 0\}.$$

Let $\rho \equiv \rho_t := e^{t\mathcal{L}_A^{\beta*}}(\rho)$ in Lemma 6.6.2 and note that the generator \mathcal{L}_A^β satisfies the following so-called *GNS detailed balance condition* (see Definition 4.2.7) with respect to the state σ for any $X, Y \in \mathcal{B}(\mathcal{H})$:

$$\text{tr}(\sigma e^{t\mathcal{L}_A^\beta}(X)^* Y) = \text{tr}(\sigma X^* e^{t\mathcal{L}_A^\beta}(Y)). \quad (11.11)$$

Then, we can use the following representation of the truncated generator \mathcal{L}_A^β of the Davies semigroup at inverse temperature β , different from that shown in Equation (11.8):

$$\mathcal{L}_A^\beta(X) = \sum_{j \in \mathcal{J}_A} e^{-\omega_j/2} \chi_j (L_j^* [X, L_j] + [L_j^*, X] L_j), \quad (11.12)$$

where given a multi-index $j = (k, \alpha, \omega) \in \mathcal{J}_A$, we have $k \in A$, $\omega_j = -\beta\omega$ and $L_j = S_{\alpha,k}(\omega)$. Remark that the operators L_j do not depend on β . This expression follows directly from the Lindblad form (11.7) after denoting

$$\chi_j \equiv \chi_j^\beta := e^{-\beta\omega} \chi_{\alpha,k}^\beta(\omega) \geq 0.$$

Recall that, for any $\beta \in \mathbb{R}$, the Lindblad operators L_j satisfy

$$\Delta_\sigma(L_j) = e^{\beta\omega} L_j = e^{-\beta\omega_j} L_j. \quad (11.13)$$

With this form of the Lindblad generator, Carlen and Maas showed that the conditional entropy production can be written as follows [CM17]:

$$\text{EP}_A(\rho) = \int_0^1 \sum_{j \in \mathcal{J}_A} \chi_j e^{\omega_j(1/2-s)} \text{tr}([L_j, (\ln \rho - \ln \sigma)]^* \rho^s [L_j, (\ln \rho - \ln \sigma)] \rho^{1-s}) ds, \quad (11.14)$$

with $[L_j, \ln \sigma] = \omega_j L_j$ arising from the eigenvector Equations (11.13). This will be of use in the next sections, mainly in Section 11.3.

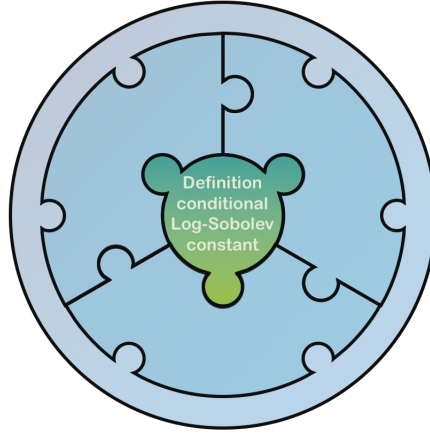


Figure 11.1: Definition of the conditional log-Sobolev constant for the Davies dynamics.

R Remark 11.1.2

Note that the essential condition to write the conditional Davies generator as in Equation (11.12), and thus prove that the conditional entropy production can be written as in Equation (11.14), is the GNS detailed balance condition. For the set of semigroups verifying this detailed balance condition, and by a non-commutative version of the Holley-Stroock perturbation argument whose existence has been communicated to us by private communication [JLR19], the conditional log-Sobolev constant can be seen to be positive following the argument developed in Section 11.2. Since the heat-bath semigroup lacks this detailed balance condition (although verifies the KMS one), this result does not follow for the heat-bath dynamics and thus some other techniques have to be explored to prove positivity of the conditional log-Sobolev constant, as shown in Chapter 10.

To conclude this section, let us introduce the conditional log-Sobolev constant for the Davies dynamics (see Figure 11.1).

Definition 11.1.3 — CONDITIONAL LOG-SOBOLEV CONST. FOR DAVIES DYNAMICS, (BCR19b).

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice and let σ_Λ be the Gibbs state of a local, commuting Hamiltonian as introduced above. Let $\mathcal{L}_\Lambda^\beta$ be the Davies generator with σ_Λ as fixed point. Given an inverse temperature $\beta > 0$, the *conditional log-Sobolev constant* of $\mathcal{L}_\Lambda^\beta$ in $A \subseteq \Lambda$ is defined as

$$\alpha_\Lambda(\mathcal{L}_A^{\beta*}) := \inf_{\rho \in \mathcal{S}_\Lambda} \frac{-\text{tr}[\mathcal{L}_A^{\beta*}(\rho)(\log \rho - \log \sigma)]}{2D_A^\mathcal{E}(\rho||\sigma)},$$

where $D_A^\mathcal{E}(\rho||\sigma)$ is the general conditional relative entropy by expectations introduced in Definition 6.6.1 for \mathcal{E} as in Equation (11.9).

R Remark 11.1.4

This constant is equal to the log-Sobolev constant of $\mathcal{L}_\Lambda^\beta$ in the classical case, due to the DLR condition (see e.g. [DPP02]). However, this property is not known to hold in the quantum case, and hence the problem of the positivity of $\alpha_\Lambda(\mathcal{L}_A^{\beta*})$ is not obvious in our setting.

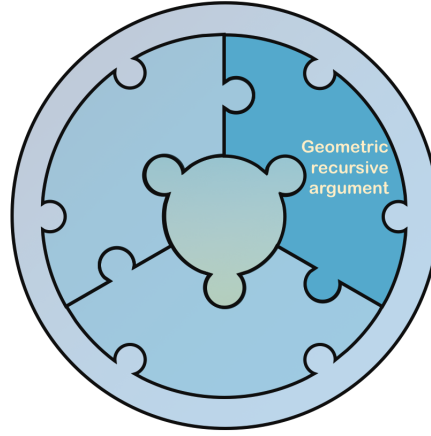


Figure 11.2: Piece for the geometric recursive argument for the Davies dynamics.

11.2 REDUCTION FROM GLOBAL TO CONDITIONAL LOG-SOBOLEV CONSTANT

In this section, we reduce the problem of the positivity of the log-Sobolev constant for the Davies generators to the one of the positivity of the conditional log-Sobolev constant for a fixed finite sublattice A . Indeed, the main result of this subsection consists of a lower bound for the log-Sobolev constant of the lattice Λ in terms of conditional log-Sobolev constants in a subregion of Λ for fixed β , which constitutes the necessary geometric recursive argument for the proof of the positivity of the log-Sobolev constant (see Figure 11.2).

The geometric construction that we devise was already used in order to prove the result in the case of classical Gibbs samplers [Ces01] [DPP02], as well as in the proof of the positivity of the spectral gap of Davies generators in [KB16]. First, we need some concepts related to the construction we are going to use in the proof of the main result. In particular, we make use of the concept of a “fat rectangle”, already introduced in Chapter 3.

Definition 11.2.1 — RECTANGLE.

Let $x \in \mathbb{Z}^d$ be a site and $l_1, \dots, l_d \in \mathbb{N}$. We can define the following *rectangle*:

$$R(x; l_1, \dots, l_d) := x + ([1, l_1] \times \dots \times [1, l_d]) \cap \mathbb{Z}^d. \quad (11.15)$$

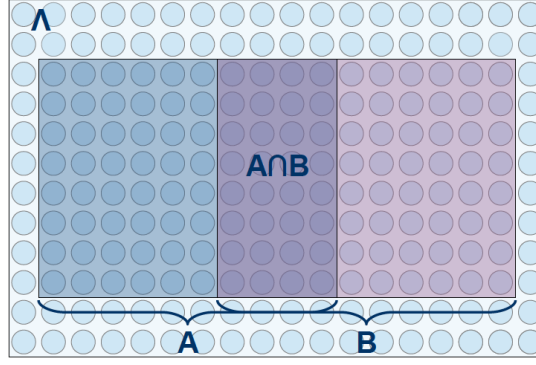
Given a rectangle of this form, we define its *size* by $\max\{l_k : k = 1, \dots, d\}$, and we say that the rectangle is *fat* if

$$\min\{l_k : k = 1, \dots, d\} \geq \frac{1}{10} \max\{l_k : k = 1, \dots, d\}. \quad (11.16)$$

Let us denote by \mathcal{R}_L the class of all fat rectangles in \mathbb{Z}^d of size at most $L \in \mathbb{N}$ and $\mathcal{R} = \bigcup_{L \geq 1} \mathcal{R}_L$.

In what follows we are interested in the lattice case where, given two overlapping subregions $A, B \subset \Lambda$, $\mathcal{E}_1 = \mathcal{E}_A^\beta$, $\mathcal{E}_2 = \mathcal{E}_B^\beta$ and $\mathcal{E}_{\mathcal{M}} = \mathcal{E}_{A \cup B}^\beta$ are the conditional expectations defined in Section 11.1.

The following theorem is the main result of this section:

Figure 11.3: Splitting in A and B .**Theorem 11.2.2 — FROM LOG-SOBOLEV TO CONDITIONAL LOG-SOBOLEV CONST., (BCR19b).**

Let $\Lambda \subseteq \mathbb{Z}^d$ and let $\Phi : \Lambda \mapsto \mathcal{A}_\Lambda$ be an r -local bounded and commuting potential. Assume that the Gibbs state σ_Λ of corresponding Hamiltonian H_Λ satisfies exponential conditional \mathbb{L}_1 -clustering of correlations as defined in Definition 8.3.8. Then, there exists an integer $L_0 > 0$ for which the following holds:

$$\alpha(\mathcal{L}_\Lambda^{\beta*}) \geq \Psi(L_0) \min_{R \in \mathcal{R}_{L_0}} \alpha_\Lambda(\mathcal{L}_R^{\beta*}),$$

where $\Psi(L_0)$ is a constant independent of the size of Λ .

Although some parts of the proof resemble those of Theorem 3.5.5 for classical spin systems, we show here a complete proof of Theorem 11.2.2 for completeness.

We will divide the proof of this result in several steps. In the first step, we lower bound the conditional log-Sobolev constant in the union of two regions (as we show in Figure 11.3) in terms of the conditional log-Sobolev constants in each one of them. This result will serve as the base step of our geometric recursive argument later.

Step 11.2.3 Assuming exponential conditional \mathbb{L}_1 -clustering of correlations, the following holds for every $\rho_\Lambda \in \mathcal{S}_\Lambda$ and $A, B \subset \Lambda$ such that $c(A, B) := c e^{-d(A \setminus B, B \setminus A)/\xi} < 2(4 + \sqrt{2})$:

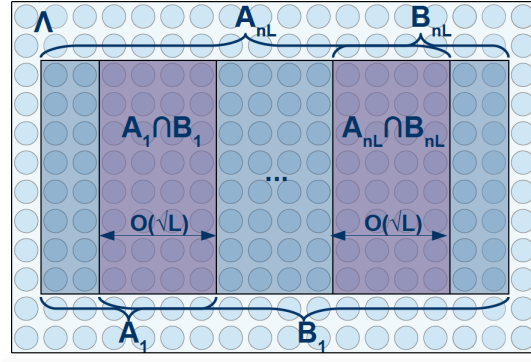
$$D(\rho_\Lambda \| \mathcal{E}_{A \cup B}^{\beta*}(\rho_\Lambda)) \leq \frac{\theta(A, B)}{2 \min \left\{ \alpha_\Lambda(\mathcal{L}_A^{\beta*}), \alpha_\Lambda(\mathcal{L}_B^{\beta*}) \right\}} (\text{EP}_{A \cap B}(\rho_\Lambda) + \text{EP}_{A \cup B}(\rho_\Lambda)),$$

$$\text{where } \theta(A, B) := \frac{1}{1 - 2(4 + \sqrt{2}) c e^{-d(A \setminus B, B \setminus A)/\xi}}.$$

Proof. First, define $\mathcal{E}_1 = \mathcal{E}_A^\beta$, $\mathcal{E}_2 = \mathcal{E}_B^\beta$ and $\mathcal{E}_\mathcal{M} = \mathcal{E}_{A \cup B}^\beta$ to be the conditional expectations defined in Section 8.3. By virtue of Corollary 8.4.6, we have for every density matrix ρ_Λ :

$$D(\rho_\Lambda \| \mathcal{E}_{A \cup B}^{\beta*}(\rho_\Lambda)) \leq \theta(A, B) \left(D(\rho_\Lambda \| \mathcal{E}_A^{\beta*}(\rho_\Lambda)) + D(\rho_\Lambda \| \mathcal{E}_B^{\beta*}(\rho_\Lambda)) \right)$$

Now, recalling the definitions of the conditional log-Sobolev constants in A and B , respec-

Figure 11.4: Splitting in A_n and B_n .

tively (see Definition 11.1.3), one has

$$\begin{aligned}
 D\left(\rho_\Lambda \parallel \mathcal{E}_{A \cup B}^{\beta*}(\rho_\Lambda)\right) &\leq \theta(A, B) \left(\frac{-\text{tr}[\mathcal{L}_A^{\beta*}(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2\alpha_\Lambda(\mathcal{L}_A^{\beta*})} + \frac{-\text{tr}[\mathcal{L}_B^{\beta*}(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2\alpha_\Lambda(\mathcal{L}_B^{\beta*})} \right) \\
 &\leq \frac{\theta(A, B)}{2 \min \left\{ \alpha_\Lambda(\mathcal{L}_A^{\beta*}), \alpha_\Lambda(\mathcal{L}_B^{\beta*}) \right\}} (\text{EP}_A(\rho_\Lambda) + \text{EP}_B(\rho_\Lambda)) \\
 &= \frac{\theta(A, B)}{2 \min \left\{ \alpha_\Lambda(\mathcal{L}_A^{\beta*}), \alpha_\Lambda(\mathcal{L}_B^{\beta*}) \right\}} (\text{EP}_{A \cap B}(\rho_\Lambda) + \text{EP}_{A \cup B}(\rho_\Lambda)),
 \end{aligned}$$

where in the last equality we are using the fact that

$$\mathcal{L}_A^{\beta*}(\rho_\Lambda) + \mathcal{L}_B^{\beta*}(\rho_\Lambda) = \mathcal{L}_{A \cup B}^{\beta*}(\rho_\Lambda) + \mathcal{L}_{A \cap B}^{\beta*}(\rho_\Lambda) \quad (11.17)$$

for every $\rho_\Lambda \in \mathcal{S}_\Lambda$. ■

In the second step of the proof, we split a certain region of the lattice into two subregions and get a lower bound for the conditional log-Sobolev constant of the former in terms of the conditional log-Sobolev constants in the latter. For that, we construct a suitable family of rectangles in Λ where we apply the previous step.

Let us first define the following quantity:

$$S(L) := \inf_{R \in \mathcal{R}_L} \alpha_\Lambda \left(\mathcal{L}_R^{\beta*} \right) \quad \forall L \geq 1. \quad (11.18)$$

Let $R := R(x; l_1, \dots, l_d)$. Without loss of generality, assume that $x = 0$, and $l_1 \leq \dots \leq l_d$. Let us also suppose that $L < l_d \leq 2L$. We define $a_L := \lfloor \sqrt{L} \rfloor$ and $n_L := \lfloor \frac{L}{10a_L} \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part. For every integer $1 \leq n \leq n_L$, we cover R with the following pair of rectangles:

$$\begin{aligned}
 A_n &:= \left\{ x \in R : 0 \leq x_d \leq \frac{l_d}{2} + na_L \right\}, \\
 B_n &:= \left\{ x \in R : \frac{l_d}{2} + (n-1)a_L < x_d \leq l_d \right\}.
 \end{aligned} \quad (11.19)$$

Hence, for n fixed, it is clear that $A_n \cap B_n \neq \emptyset$ and the shortest side of the overlap has length of order \sqrt{L} (due to the fact that we are considering R a fat rectangle, so $l_1 \geq \frac{1}{10} l_d > \frac{L}{10}$ and if we had $\sqrt{L} > l_1$, we would have $\sqrt{L} > \frac{L}{10}$, or, equivalently, $\frac{L}{100} < 1$, which only holds for L small). See Figure 11.4.

Step 11.2.4 There exists a positive constant C , independent of the size of $L < l_d \leq 2L$ of R such that

$$\min_{n=1, \dots, n_L} \left\{ \alpha_\Lambda \left(\mathcal{L}_{A_n}^{\beta*} \right), \alpha_\Lambda \left(\mathcal{L}_{B_n}^{\beta*} \right) \right\} \left(1 + \frac{C}{\sqrt{L}} \right)^{-1} \leq \alpha_\Lambda \left(\mathcal{L}_R^{\beta*} \right), \quad (11.20)$$

for every $1 \leq n \leq n_L$ and L large enough.

Proof. If we use the sets defined in Equation (11.19) in the expression obtained in Step 11.2.3, we get, for every $1 \leq n \leq n_L$ and for every $\rho_\Lambda \in \mathcal{S}_\Lambda$,

$$D \left(\rho_\Lambda \left\| \mathcal{E}_R^{\beta*}(\rho_\Lambda) \right\| \right) \leq \frac{\theta(A_n, B_n)}{2 \min \left\{ \alpha_\Lambda(\mathcal{L}_{A_n}^{\beta*}), \alpha_\Lambda(\mathcal{L}_{B_n}^{\beta*}) \right\}} (\text{EP}_{A_n \cap B_n}(\rho_\Lambda) + \text{EP}_{A_n \cup B_n}(\rho_\Lambda)), \quad (11.21)$$

where

$$\theta(A_n, B_n) = \frac{1}{1 - 2(4 + \sqrt{2})c e^{-\sqrt{L}/\xi}}$$

for every $1 \leq n \leq n_L$. Let us denote the latter by $\theta(\sqrt{L})$. Now, by the definition of A_n and B_n , the two following properties clearly hold:

1. $A_i \cap B_i \cap A_j \cap B_j = \emptyset$ for every $i \neq j$;
2. $\bigcup_{1 \leq n \leq n_L} (A_n \cap B_n) \subseteq R$.

Therefore, we can average over n the previous expression to obtain:

$$\begin{aligned} D \left(\rho_\Lambda \left\| \mathcal{E}_R^{\beta*}(\rho_\Lambda) \right\| \right) &\leq \frac{1}{n_L} \sum_{n=1}^{n_L} \frac{\theta(A_n, B_n)}{2 \min \left\{ \alpha_\Lambda(\mathcal{L}_{A_n}^{\beta*}), \alpha_\Lambda(\mathcal{L}_{B_n}^{\beta*}) \right\}} (\text{EP}_{A_n \cap B_n}(\rho_\Lambda) + \text{EP}_R(\rho_\Lambda)) \\ &\leq \frac{\theta(\sqrt{L})}{2 \min_{n=1, \dots, n_L} \left\{ \alpha_\Lambda(\mathcal{L}_{A_n}^{\beta*}), \alpha_\Lambda(\mathcal{L}_{B_n}^{\beta*}) \right\}} \left(\text{EP}_R(\rho_\Lambda) + \frac{1}{n_L} \sum_{n=1}^{n_L} \text{EP}_{A_n \cap B_n}(\rho_\Lambda) \right) \\ &\leq \frac{\theta(\sqrt{L})}{2 \min_{n=1, \dots, n_L} \left\{ \alpha_\Lambda(\mathcal{L}_{A_n}^{\beta*}), \alpha_\Lambda(\mathcal{L}_{B_n}^{\beta*}) \right\}} \left(1 + \frac{1}{n_L} \right) \text{EP}_R(\rho_\Lambda). \end{aligned}$$

Hence, by the definition of $\alpha_\Lambda(\mathcal{L}_R^{\beta*})$, we have

$$\frac{\min_{n=1, \dots, n_L} \left\{ \alpha_\Lambda(\mathcal{L}_{A_n}^{\beta*}), \alpha_\Lambda(\mathcal{L}_{B_n}^{\beta*}) \right\}}{\theta(\sqrt{L})} \left(1 + \frac{1}{n_L} \right)^{-1} \leq \alpha_\Lambda(\mathcal{L}_R^{\beta*}), \quad (11.22)$$

Note that

$$\theta(\sqrt{L}) \geq 1 \text{ for every } L > 1 \text{ and } \lim_{L \rightarrow \infty} \theta(\sqrt{L}) = 1.$$

Then, for L large enough, the following inequality holds:

$$\min_{n=1, \dots, n_L} \left\{ \alpha_\Lambda(\mathcal{L}_{A_n}^{\beta*}), \alpha_\Lambda(\mathcal{L}_{B_n}^{\beta*}) \right\} \left(1 + \frac{C}{\sqrt{L}} \right)^{-1} \leq \alpha_\Lambda(\mathcal{L}_R^{\beta*}), \quad (11.23)$$

for $C > 1$ independent of L . ■

Let us denote by L_0 the first integer for which inequality (11.23) holds. We will use it in the last step of the proof. First, we obtain a recursion between the quantities $S(L)$ which will allow us to get a lower bound for the global log-Sobolev constant in terms of size-fixed conditional log-Sobolev constants.

Step 11.2.5 There exists a positive constant K independent of the size of R such that

$$S(2L) \geq \left(1 + \frac{K}{\sqrt{L}}\right)^{-3d} S(L) \quad \text{for } L \text{ large enough.} \quad (11.24)$$

Proof. Consider the expression obtained in the previous step. Let us analyze the value of the log-Sobolev constant in the rectangles A_n and B_n .

Let us consider the rectangle A_n (the analysis is analogous for B_n). We can write it as

$$A_n := x_{A_n} + \left([1, l_1] \times \dots \times [1, l_{d-1}] \times \left[1, \frac{l_d}{2} + na_L\right] \right) \cap \mathbb{Z}^d \quad (11.25)$$

The side corresponding to the coordinate x_d has length less than or equal to $1.2L$, by the definition of A_n . For the other sides, we have to distinguish between two different cases.

1. If $\max \{l_k : k = 1, \dots, d-1\} \leq \frac{3}{2}L$, then the longest side of A_n is less than or equal to $\frac{3}{2}L$, so $A_n \in \mathcal{R}_{\frac{3}{2}L}$ and $\alpha_\Lambda(\mathcal{L}_{A_n}^{\beta*}) \geq S\left(\frac{3}{2}L\right)$.
2. If the greatest side of A_n , which we call l_i , satisfies $l_i > \frac{3}{2}L$, it is clear that A_n verifies $\max\{l_k\} > 1.5L$ and $\min\{l_k\} \leq 1.2L$. Hence,

$$\alpha_\Lambda(\mathcal{L}_{A_n}^{\beta*}) \geq \min_{R: \max\{l_k\} > 1.5L, \min\{l_k\} \leq 1.2L} \alpha_\Lambda(\mathcal{L}_R^{\beta*}). \quad (11.26)$$

Therefore, for the right-hand side of Equation (11.20), we have

$$\begin{aligned} & \left(1 + \frac{C}{\sqrt{L}}\right)^{-1} \min_{n=1, \dots, n_L} \left\{ \alpha_\Lambda(\mathcal{L}_{A_n}^{\beta*}), \alpha_\Lambda(\mathcal{L}_{B_n}^{\beta*}) \right\} \\ & \geq \left(1 + \frac{C}{\sqrt{L}}\right)^{-1} \min \left\{ S\left(\frac{3}{2}L\right), \min_{R: \max\{l_k\} > 1.5L, \min\{l_k\} \leq 1.2L} \alpha_\Lambda(\mathcal{L}_R^{\beta*}) \right\}. \end{aligned}$$

Now, we consider a rectangle in \mathcal{R}_{2L} such that its longest side is greater than or equal to $1.5L$ and its shortest side has length less than or equal to $1.2L$. Iterating Step 11.2.4 at most $d-1$ times on that rectangle, we end up with a rectangle whose longest side is shorter than or equal to $1.5L$. Hence,

$$\min_{R: \max\{l_k\} > 1.5L, \min\{l_k\} \leq 1.2L} \alpha_\Lambda(\mathcal{L}_R^{\beta*}) \geq \left(1 + \frac{C}{\sqrt{L}}\right)^{-(d-1)} S\left(\frac{3}{2}L\right). \quad (11.27)$$

Therefore,

$$\left(1 + \frac{C}{\sqrt{L}}\right)^{-1} \min_{n=1, \dots, n_L} \left\{ \alpha_\Lambda(\mathcal{L}_{A_n}^{\beta*}), \alpha_\Lambda(\mathcal{L}_{B_n}^{\beta*}) \right\} \geq \left(1 + \frac{C}{\sqrt{L}}\right)^{-d} S\left(\frac{3}{2}L\right), \quad (11.28)$$

and since the rectangle that we were considering in Step 11.2.4 verified $R \in \mathcal{R}_{2L}$, we obtain

$$S(2L) \geq \left(1 + \frac{C}{\sqrt{L}}\right)^{-d} S\left(\frac{3}{2}L\right). \quad (11.29)$$

To conclude, we iterate this expression two more times to obtain

$$S(2L) \geq \left(1 + \frac{C}{\sqrt{L}}\right)^{-d} \left(1 + \frac{C}{\sqrt{\frac{3L}{4}}}\right)^{-d} \left(1 + \frac{C}{\sqrt{\frac{9L}{16}}}\right)^{-d} S\left(\frac{27}{32}L\right), \quad (11.30)$$

and since $S\left(\frac{27}{32}L\right) \geq S(L)$, we obtain

$$S(2L) \geq \left(1 + \frac{K}{\sqrt{L}}\right)^{-3d} S(L), \quad (11.31)$$

where K is a constant independent of the size of the system. ■

Finally, in the last step of the proof, using recursively the relation obtained in the previous one, we get a lower bound for the global log-Sobolev constant in terms of conditional log-Sobolev constants.

Step 11.2.6 There exists a constant $L_0 \in \mathcal{N}$, independent of Λ such that the following holds:

$$\alpha\left(\mathcal{L}_\Lambda^{\beta*}\right) \geq \Psi(L_0) S(L_0),$$

where $\Psi(L_0)$ does not depend on the size of Λ .

Proof. By virtue of the previous step, it is clear that the following holds for L_0 as defined above:

$$S(2L_0) \geq \left(1 + \frac{K}{\sqrt{L_0}}\right)^{-3d} S(L_0), \quad (11.32)$$

Note now that the limit of Λ tending to \mathbb{Z}^d is the same as the one of $S(nL_0)$ with n tending to infinity. Therefore,

$$\begin{aligned} \lim_{\Lambda \rightarrow \mathbb{Z}^d} \alpha_\Lambda\left(\mathcal{L}_\Lambda^{\beta*}\right) &= \lim_{n \rightarrow \infty} S(2^n L_0) \\ &\geq \left(\prod_{n=1}^{\infty} \left(1 + \frac{K}{\sqrt{2^{n-1}L_0}}\right)\right)^{-3d} S(L_0) \\ &\geq \left(\exp \left[\sum_{n=0}^{\infty} \frac{K}{2^n L_0}\right]\right)^{-3d} S(L_0) \\ &= \exp \left[\frac{-3dK}{L_0} (2 + \sqrt{2})\right] S(L_0), \end{aligned}$$

where the constants L_0 and K do not depend on the size of Λ . ■

The previous reduction from the global log-Sobolev constant to the conditional one for the Davies dynamics via quasi-factorization of the relative entropy (Theorem 8.4.1) has been proven under the assumption of exponential conditional \mathbb{L}_1 -clustering of correlations. Let us recall that an analogous result to the aforementioned theorem of quasi-factorization has also been proven in Chapter 8 under the assumption of covariance-entropy clustering of correlations (Theorem 8.5.7). Then, we also have the following result for that assumption.

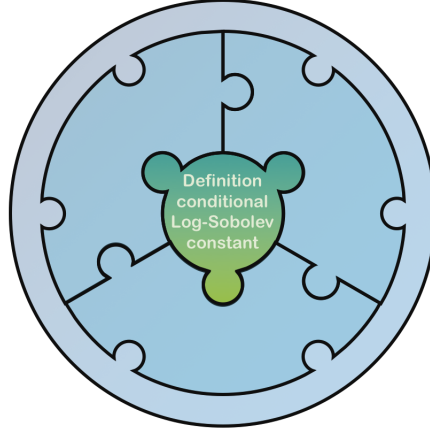


Figure 11.5: Piece concerning the positivity of the conditional log-Sobolev constant.

Theorem 11.2.7 — FROM LOG-SOBOLEV TO CONDITIONAL LOG-SOBOLEV CONSTANT (2).

Let $\Lambda \subseteq \mathbb{Z}^d$ and let $\Phi : \Lambda \mapsto \mathcal{A}_\Lambda$ be an r -local bounded and commuting potential. Assume that the Gibbs state σ_Λ of corresponding Hamiltonian H_Λ satisfies covariance-entropy clustering of correlations as defined in Definition 8.5.5. Then, there exists an integer $L_0 > 0$ for which the following holds:

$$\alpha \left(\mathcal{L}_\Lambda^{\beta*} \right) \geq \Psi(L_0) \min_{R \in \mathcal{R}_{L_0}} \alpha_\Lambda \left(\mathcal{L}_R^{\beta*} \right),$$

where $\Psi(L_0)$ is a constant independent of the size of Λ .

11.3 DISCUSSION ON THE POSITIVITY OF THE LOG-SOBOLEV CONSTANT

In this section, we put the previously proven pieces of the puzzle together with the missing one, namely the proof of positivity of conditional log-Sobolev constant for the Davies dynamics (see Figure 11.5), to conclude the discussion on the positivity of the global one.

First, let us recall that we are considering the definition for the conditional log-Sobolev constant presented in Definition 11.1.3, where the conditional relative entropy we consider is the general conditional relative entropy by expectations (see Definition 6.6.1) for the conditional expectation associated to the Davies dynamics, Equation (11.9). Now, assuming on the invariant states of the conditional dynamics either conditional \mathbb{L}_1 -clustering of correlations (Condition 8.3.3) or covariance-entropy clustering of correlations (Condition 8.5.5), we have proven two completely analogous results of strong quasi-factorization of the relative entropy, Theorem 8.4.1 and Theorem 8.5.7, respectively.

Subsequently, in Section 11.2, we have reduced the problem of proving positivity of the log-Sobolev constant for the Davies dynamics to proving positivity for the conditional one, via the results of strong quasi-factorization of the relative entropy mentioned above. To conclude, the only part left is the proof of the positivity of the conditional log-Sobolev constant. We can pose that as the following conjecture.

Conjecture 11.3.1 — POSITIVITY OF THE CONDITIONAL LOG-SOBOLEV CONSTANT.

Given $\Lambda \subset \mathbb{Z}^d$, $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \rightarrow \mathcal{S}_\Lambda$ the Lindbladian associated to the Davies dynamics and a finite lattice and $A \subset \Lambda$, we have

$$\alpha_\Lambda(\mathcal{L}_A^{\beta*}) \geq \psi(|A|) > 0,$$

where $\psi(|A|)$ might depend on Λ , but is independent of its size.

This conjecture leads to the following result.

Theorem 11.3.2 — LOG-SOBOLEV CONSTANT FOR THE DAVIES DYNAMICS, (BCR19b).

Let $\Lambda \subset \mathbb{Z}^d$ be a finite lattice and let β be a finite inverse temperature. Consider $\mathcal{L}_\Lambda^{\beta*} : \mathcal{S}_\Lambda \rightarrow \mathcal{S}_\Lambda$ the Lindbladian associated to the Davies dynamics and assume that either conditional \mathbb{L}_1 -clustering of correlations or covariance-entropy clustering of correlations is satisfied. Then, if Conjecture 11.3.1 holds true, $\mathcal{L}_\Lambda^{\beta*}$ has a positive log-Sobolev constant which is independent of $|\Lambda|$.

R Remark 11.3.3

Conjecture 11.3.1 can be proven based on a non-commutative and non-primitive version of the Holley-Stroock perturbation principle [HS87], a result that has been communicated to us by private communication to have been recently proven [JLR19], but it is not published yet.

11.4 EXAMPLE

The aim of this section is to show an example of a system satisfying the covariance-entropy clustering of correlations. For that, we investigate a quantum lattice spin system undergoing a classical Glauber dynamics, whose framework was already studied in [Cub+15].

First, let us introduce the generator. Consider a lattice spin system over $\Gamma = \mathbb{Z}^d$ with classical configuration space $S = \{+1, -1\}$, and, for each $\Lambda \subset \Gamma$, denote by $\Omega_\Lambda = S^\Lambda$ the space of configurations over Λ . Analogously to what we showed in Chapter 3, given a classical finite-range, translationally invariant potential $\{\Phi_A\}_{A \in \Gamma}$ and a boundary condition $\tau \in \Omega_{\Lambda^c}$, we define the Hamiltonian over Λ as

$$H_\Lambda^\tau(\sigma) = - \sum_{A \cap \Lambda \neq \emptyset} J_A(\sigma \times \tau), \quad \forall \sigma \in \Omega_\Lambda.$$

The classical Gibbs state corresponding to such Hamiltonian is then given by

$$\mu_\Lambda^\tau(\sigma) = (Z_\Lambda^\tau)^{-1} \exp(-H_\Lambda^\tau(\sigma)),$$

Now, we define the Glauber dynamics for a potential Φ as the Markov process on Ω_Λ with the generator

$$(L_\Lambda f)(\sigma) = \sum_{x \in \Lambda} c_\Phi(x, \sigma) \nabla_x f(\sigma),$$

where $\nabla_x f(\sigma) = f(\sigma^x) - f(\sigma)$ and σ^x is the configuration obtained by flipping the spin at position x . The numbers $c_\Phi(x, \sigma)$ are called transition rates and must satisfy the following assumptions:

1. There exist c_m, c_M such that $0 < c_m \leq c_\Phi(x, \sigma) \leq c_M < \infty$ for all x, σ .

2. $c_\Phi(x, \cdot)$ depends only on spin values in $b_r(x)$.
3. For all $k \in \Gamma$, $c_\Phi(x, \sigma') = c_\Phi(x+k, \sigma)$ id $\sigma'(y) = \sigma(y+k)$ for all y .
4. Detailed balance: for all $x \in \Gamma$, and all σ

$$\exp\left(-\sum_{A \ni x} \Phi_A(\sigma)\right) c_\Phi(x, \sigma) = c_\Phi(x, \sigma^x) \exp\left(-\sum_{A \ni x} J_A(\sigma^x)\right).$$

These assumptions constitute sufficient conditions for the corresponding Markov process to have the Gibbs states over Λ as stationary points. We can now introduce the notion of a quantum embedding of the aforementioned classical Glauber dynamics. This is the Lindbladian of corresponding Lindblad operators given by

$$L_{x,\eta} := \sqrt{c_J(x, \eta)} |\eta^x\rangle \langle \eta| \otimes \mathbb{1}, \quad \forall x \in \Lambda, \eta \in \Omega_{b_x(r)}. \quad (11.33)$$

It was shown in [Cub+15] that such a dynamics is KMS-symmetric with respect to the state μ_Λ^τ as embedded into the computational basis. Moreover, the set of fixed points is equal to the convex hull of the set of Gibbs states over Λ , $\{\mu_\Lambda^\tau | \tau \in \Omega_{\Lambda^c}\}$.

To show that the classical Glauber dynamics satisfies the covariance-entropy clustering of correlations, let us take an observable $X \in M_{\text{diag}}(\mathcal{H}_\Lambda)$, that is, diagonal in the computation basis. Then, it decomposes as follows:

$$X = \sum_{\omega \in \Omega_{(A \cup B)^c}} |\omega\rangle \langle \omega|_{(A \cup B)^c} \otimes X_{AB}^\omega.$$

We need to bound the term $\langle \mathcal{E}_A(X) - \mathcal{E}_{A \cup B}(X), \mathcal{E}_B(X) - \mathcal{E}_{A \cup B}(X) \rangle_{\mu_{AB}}$ by the relative entropy $D(\mu_{AB} X \| \mu_{AB})$, where $\mathcal{E}_{A \cup B}^*(\mu_{AB}) = \mu_{AB}$ and $\mathcal{E}_{A \cup B}(X) = \mathbb{1}$. From that last identity, and the ‘‘DLR’’ decomposition of μ_{AB} :

$$\mu_{AB} = \sum_{\omega \in \Omega_{(A \cup B)^c}} p_{AB}(\omega) |\omega\rangle \langle \omega|_{(A \cup B)^c} \otimes \sigma_{A \cup B}^\omega,$$

where $\sigma_{A \cup B}^\omega$ is a Gibbs state over $A \cup B$. Then, we have,

$$\mathcal{E}_{A \cup B}(X) = \mathbb{1} = \sum_{\omega \in \Omega_{(A \cup B)^c}} |\omega\rangle \langle \omega|_{(A \cup B)^c} \otimes \mathbb{1}_{A \cup B} \text{tr}[X_{AB}^\omega \sigma_{A \cup B}^\omega],$$

and thus

$$\text{tr}[X_{AB}^\omega \sigma_{A \cup B}^\omega] = 1 \quad \forall \omega \in \Omega_{(A \cup B)^c}.$$

Using this, together with the fact that, with a slight abuse of notations, $\mathcal{E}_A = \mathcal{E}_A \otimes \text{Id}_{A^c}$, we can compute the covariance:

$$\begin{aligned} & \langle \mathcal{E}_A(X) - \mathbb{1}, \mathcal{E}_B(X) - \mathbb{1} \rangle_{\mu_{AB}} \\ &= \sum_{\omega \in \Omega_{(A \cup B)^c}} p_{AB}(\omega) \langle X_{AB}^\omega - \mathbb{1}_{A \cup B}, \mathcal{E}_A \circ \mathcal{E}_B(X_{AB}^\omega) - \mathbb{1}_{A \cup B} \rangle_{\sigma_{A \cup B}^\omega} \\ &\leq \sum_{\omega \in \Omega_{(A \cup B)^c}} p_{AB}(\omega) \|X_{AB}^\omega - \mathbb{1}_{A \cup B}\|_{\mathbb{L}_1(\sigma_{A \cup B}^\omega)} \|\mathcal{E}_A \circ \mathcal{E}_B(X_{A \cup B}^\omega) - \mathbb{1}_{A \cup B}\|_{\mathbb{L}_\infty(\sigma_{A \cup B}^\omega)}, \end{aligned}$$

where we used Hölder’s inequality in the inequality. Now, using the weak exponential clustering assumption of [Ces01], which is a condition that we now that the classical Glauber dynamics satisfies, in order to further bound the \mathbb{L}_∞ norm, we obtain:

$$\|\mathcal{E}_A \circ \mathcal{E}_B(X_{A \cup B}^\omega) - \mathbb{1}_{A \cup B}\|_{\mathbb{L}_\infty(\sigma_{A \cup B}^\omega)} \leq c e^{-d(A \setminus B, B \setminus A)/\xi} \|X_{AB}^\omega - \mathbb{1}_{A \cup B}\|_{\mathbb{L}_1(\sigma_{A \cup B}^\omega)}$$

We conclude the proof of the covariance-entropy clustering of correlations using Pinsker’s inequality and the data processing inequality.



SUMMARY OF RESULTS

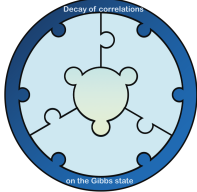
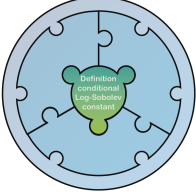
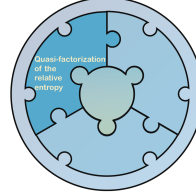
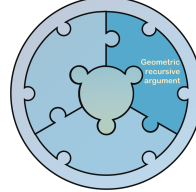
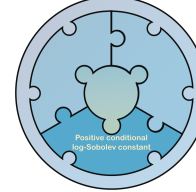
To conclude Part III, we summarize below all the results of concerning positivity of log-arithmetic Sobolev constants developed in the previous chapters. Before that, let us recall that, given a finite lattice $\Lambda \subset \mathbb{Z}^d$, \mathcal{H}_Λ its associated Hilbert space and $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \rightarrow \mathcal{S}_\Lambda$ a primitive, reversible Lindbladian with fixed point $\sigma_\Lambda \in \mathcal{S}_\Lambda$, the *log-Sobolev constant* of \mathcal{L}_Λ^* is given by

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\text{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}.$$

Moreover, given a quantum system, the strategy devised to prove that such system has a positive log-Sobolev constant consists of the following five steps:

1. **Definition.** Definition of some *clustering conditions* on the Gibbs state.
2. **Definition.** Definition of a *conditional log-Sobolev constant*.
3. **Result.** *Quasi-factorization of the relative entropy* in terms of a conditional relative entropy.
4. **Result.** *Recursive geometric argument* to reduce the global log-Sobolev constant to the conditional one in a fixed-sized region.
5. **Result.** *Positivity of the conditional log-Sobolev constant*.

For the three different settings addressed in the previous three chapters, respectively, each one of these steps appears on the results collected in the following table.

					
DYNAMICS	CLUSTERING CONDITIONS	DEFINITION LOG-SOBOLEV	QUASI- FACTORIZATION	GEOMETRIC RECURSION	POSITIVITY COND. LOG-SOB.
Heat-bath	$\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$	Definition 9.1.1	Theorem 7.2.1	Theorem 9.1.5	Lemma 9.1.4
Heat-bath	1D systems Assumption 10.3.1 Assumption 10.3.2	Definition 10.1.3	Theorem 7.3.1	Steps 1-3 of Theorem 10.3.3	Step 4 of Theorem 10.3.3
Davies	Condition 8.3.8 or Condition 8.5.5	Definition 11.1.3	Theorem 8.5.7 or Theorem 8.4.1	Theorem 11.2.2	Conjecture 11.3.1

Part IV

DATA PROCESSING INEQUALITY

On the data processing inequality:

A strengthened DPI for the Belavkin-Staszewski relative entropy

The data processing inequality for a certain quantity in the context of quantum information theory states that this quantity cannot increase under the application of a quantum channel. Hence, the difference between that quantity before and after applying a quantum channel is always non-negative. Its applications in information theory are numerous, for instance to measure the amount of information lost when using a certain communication channel, and hence its study is fundamental.

In the second part of this thesis, we introduced the notion of *conditional relative entropy* in a subsystem as a quantity satisfying certain axioms and showed its unique form as a difference of relative entropies. This concept is necessary, in particular, for the definition of conditional log-Sobolev constant (one of the steps in the strategy discussed in Section 1.2 to prove positivity of log-Sobolev constants). A better understanding of this quantity is thus essential to improve the aforementioned strategy.

By virtue of the data processing inequality, the conditional relative entropy in a subsystem is always non-negative. However, as we will discuss in the next section, some better lower bounds for the conditional relative entropy can be found, namely non-negative quantities that lower bound the conditional relative entropy and vanish at the same states than the latter. These inequalities constitute examples of the so-called *strengthened data processing inequality for the relative entropy*.

Another quantity in the same spirit that appears frequently in the literature is the *Belavkin-Staszewski relative entropy* (*BS-entropy* for short). Given two density matrices, their BS-entropy always constitutes an upper bound for their relative entropy. Moreover, the BS-entropy also satisfies a data processing inequality, yielding the possibility to study strengthened versions of it. This is exactly the aim of the next chapter.

OVERVIEW ON THE DPI AND f -DIVERGENCES

Quantum f -divergences are important in quantum information theory, because they can be used to quantify the similarity of quantum states. Therefore, they fulfill fundamental properties such as data processing, since the distinguishability of quantum states cannot increase under the application of a quantum channel. The most important such f -divergence is the relative entropy, which we recall that is defined as

$$D(\sigma\|\rho) := \text{tr}[\sigma(\log \sigma - \log \rho)]$$

for positive definite quantum states σ, ρ . The relative entropy is one example of the so-called *standard f -divergences* [HM17, Section 3.2], which are defined as

$$S_f(\sigma\|\rho) := \text{tr}\left[\rho^{1/2}f(L_\sigma R_{\rho^{-1}})(\rho^{1/2})\right]$$

for an operator convex function $f : (0, \infty) \rightarrow \mathbb{R}$. Here, the reader should remember that L_A and R_A denote the left and right multiplication by the matrix A , respectively. The relative entropy arises by letting $f(x) = x \log x$. However, this is not the only way to generalize the classical f -divergences introduced in [AS66; Csi67]. The *maximal f -divergences* (introduced in [PR98]) are defined as

$$\hat{S}_f(\sigma\|\rho) := \text{tr}\left[\rho f(\rho^{-1/2}\sigma\rho^{-1/2})\right]$$

for an operator convex function $f : (0, \infty) \rightarrow \mathbb{R}$. They were recently studied in [Mat10] where also the name was introduced (see also [HM17, Section 3.3] and references therein).

For $f(x) = x \log x$, we obtain after a short computation the main character of this chapter, i.e. the relative entropy introduced by Belavkin and Staszewski in [BS82], which we will call *BS-entropy* for short:

$$\hat{S}_{\text{BS}}(\sigma \| \rho) := -\text{tr} \left[\sigma \log \left(\sigma^{-1/2} \rho \sigma^{-1/2} \right) \right].$$

It is known that both the standard and maximal f -divergences satisfy data processing, i.e. they decrease under the application of quantum channels. Moreover, the study of conditions for equality in the data processing inequality for the relative entropy, i.e. for which ρ, σ we have $D(\sigma \| \rho) = D(\Phi(\sigma) \| \Phi(\rho))$ for some fixed quantum channel Φ , has led to the discovery of quantum Markov states [Hay+04].

In particular, the relative entropy (and all standard f -divergences for which f is “complicated enough”) is preserved if and only if σ and ρ can be recovered by the Petz recovery map $\mathcal{R}_{\Phi}^{\rho}(X) = \rho^{1/2} \Phi^*(\Phi(\rho)^{-1/2} X \Phi(\rho)^{-1/2}) \rho^{1/2}$, i.e. $\sigma = \mathcal{R}_{\Phi}^{\rho}(\Phi(\sigma))$ and $\rho = \mathcal{R}_{\Phi}^{\rho}(\Phi(\rho))$ [Pet03]. We refer the reader to [HM17, Theorem 3.18] for a larger list of equivalent conditions. For $\Phi = \mathcal{E}$ and \mathcal{E} the trace-preserving conditional expectation onto a unital matrix subalgebra \mathcal{N} of $\mathcal{B}(\mathcal{H})$, [CV17] shows that the equality condition is stable in the sense that

$$D(\sigma \| \rho) - D(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}}) \geq \left(\frac{\pi}{8} \right)^4 \|L_{\rho} R_{\sigma^{-1}}\|_{\infty}^{-2} \|\mathcal{R}_{\Phi}^{\sigma}(\rho_{\mathcal{N}}) - \rho\|_1^4, \quad (11.34)$$

where we have written $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$ and $\rho_{\mathcal{N}} := \mathcal{E}(\rho)$. This can also be interpreted as a strengthening of the data processing inequality. Subsequent work has generalized the above result to more general standard f -divergences [CV18] and Holevo’s just-as-good fidelity [Wil18].

The difference of relative entropies that appears on the left-hand side of Equation (11.34) has been studied intensively in the context of quantum information and quantum thermodynamics [FBB18; FR18]. Moreover, for \mathcal{E} a partial trace, it has been characterized as a conditional relative entropy in Chapter 6. Equation (11.34) is the first strengthening of the data processing inequality for the relative entropy in terms of the “distance” between a state and its recovery by the Petz map, although there have been many other results with a similar spirit in the last years.

The first one of these results was presented in [FR15] and concerns the particular case of a tripartite Hilbert space \mathcal{H}_{ABC} and two dependent positive matrices ρ_{ABC} and σ_{ABC} , in the sense that $\rho_{ABC} = I_A \otimes \sigma_{BC}$. Note that the conditional mutual information is given by

$$I_{\sigma}(A : C|B) := D(\sigma_{ABC} \| \rho_{ABC}) - D(\sigma_{AB} \| \rho_{AB}),$$

where the second term in the difference corresponds to the application of the quantum channel $\mathcal{T}(\cdot) = \text{tr}_C[\cdot]$ to the first one.

Hence, in this setting, it was proven in [FR15] that the following inequality holds:

$$I_{\sigma}(A : C|B) \geq \inf_{\eta_{ABC}} (-2 \log_2 F(\sigma_{ABC}, \eta_{ABC})),$$

where

$$F(\sigma_{ABC}, \eta_{ABC}) := \|\sqrt{\sigma_{ABC}} \sqrt{\eta_{ABC}}\|_1$$

is the *fidelity* between two quantum states. More specifically, there exist unitary operations \mathcal{U}_B and \mathcal{V}_{BC} with respective unitary matrices U_B and V_{BC} on $\mathcal{H}_B, \mathcal{H}_{BC}$, respectively, such that if we consider $\mathcal{V}_{BC} \circ \mathcal{R}_{\text{tr}_C}^{\sigma_{BC}} \circ \mathcal{U}_B$, then

$$\mathcal{V}_{BC} \circ \mathcal{R}_{\text{tr}_C}^{\sigma_{BC}} \circ \mathcal{U}_B(\sigma_{AB}) = V_{BC} \sigma_{BC}^{1/2} \sigma_B^{-1/2} U_B \sigma_{AB} U_B^* \sigma_B^{-1/2} \sigma_{BC}^{1/2} V_{BC}^*,$$

we have

$$I_\sigma(A : C|B) \geq -2 \log_2 F(\sigma_{ABC}, \mathcal{V}_{BC} \circ \mathcal{R}_{\text{tr}_C}^{\sigma_{BC}} \circ \mathcal{U}_B(\sigma_{AB})),$$

where $\mathcal{V}_{BC} \circ \mathcal{R}_{\text{tr}_C}^{\sigma_{BC}} \circ \mathcal{U}_B$ is a rotated version of the Petz recovery map for the partial trace in C .

This result was subsequently lifted to obtain several other lower bounds for the difference of relative entropies, such as

$$D(\sigma||\rho) - D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)) \geq (1), (2), (3)$$

$$(1) := - \int \beta_0(t) \log F\left(\sigma, \mathcal{R}_{\mathcal{T}}^{\rho, [t]} \circ \mathcal{T}(\sigma)\right) dt \quad [\text{Jun+18}],$$

with

$$\mathcal{R}_{\mathcal{T}}^{\rho, [t]}(\cdot) = \rho^{\frac{1+i}{2}} \mathcal{T}^* \left(\mathcal{T}(\rho)^{\frac{-1-i}{2}}(\cdot) \mathcal{T}(\rho)^{\frac{-1+i}{2}} \right) \rho^{\frac{1-i}{2}}$$

and

$$\beta_0(t) = \frac{\pi}{2} (\cosh(\pi t) + 1)^{-1}.$$

$$(2) := D_M \left(\sigma \left\| \int \beta_0(t) \mathcal{R}_{\mathcal{T}}^{\sigma, [t]} \circ \mathcal{T}(\sigma) dt \right. \right) \quad [\text{SBT17}],$$

with

$$D_M(\sigma||\rho) = \sup_{(\xi, M)} D(P_{\sigma, M}||P_{\rho, M}), \text{ for } M \text{ a POVM on the power-set of a finite } \xi.$$

$$(3) := \limsup_{n \rightarrow \infty} \frac{1}{n} D \left(\sigma^{\otimes n} \left\| \int \beta_0(t) \left(\mathcal{R}_{\mathcal{T}}^{\sigma, [t]} \circ \mathcal{T}(\sigma) \right)^{\otimes n} dt \right. \right) \quad [\text{BBH17}].$$

Those results give rise to the natural question whether the difference of relative entropies can be lower bounded in terms of $D(\rho||\mathcal{R}_{\mathcal{T}}^{\rho} \circ \mathcal{T}(\sigma))$. This question can be answered negatively, as some numerical counterexamples appearing in [Bra+15b] and [FF18] show for the setting of a tripartite Hilbert space \mathcal{H}_{ABC} and two positive matrices σ_{ABC} and $\rho_{ABC} = I_A \otimes \sigma_{BC}$. Moreover, in [SR18], it is shown, again for this setting, that the latter question can be answered positively by adding a term:

$$D(\sigma_{ABC}||\mathcal{R}_{\text{tr}_C}^{\sigma_{BC}} \circ \text{tr}_C[(\sigma_{ABC})]) + \Lambda_{\max}(\sigma_{AB}||\mathcal{R}_{B \rightarrow B}) \geq I_\sigma(A : C|B),$$

where $\Lambda_{\max}(\sigma||\mathcal{E})$ is defined as the infimum over invariant states τ of \mathcal{E} of the quantity $D_{\max}(\rho||\tau)$, it verifies

$$\Lambda_{\max}(\sigma||\mathcal{E}) = 0 \Leftrightarrow \mathcal{E}(\sigma) = \sigma,$$

and

$$\mathcal{R}_{B \rightarrow B} := \text{tr}_C \circ \mathcal{R}_{\text{tr}_C}^{\sigma_{BC}}.$$

Next chapter gives analogous results to the ones in [CV17; CV18] for maximal f -divergences. For these, preservation of the maximal f -divergence, i.e. $\hat{S}_f(\Phi(\sigma)||\Phi(\rho)) = \hat{S}_f(\sigma||\rho)$, is not equivalent to σ, ρ being recoverable in the sense of Petz, although the latter implies the former. Equivalent conditions to the preservation of a maximal f -divergence for the case in which Φ is a completely positive trace-preserving map are given in [HM17, Theorem 3.34]. In our work, we prove two other equivalent conditions, which we use to prove a strengthened data processing inequality for some maximal f -divergences and in particular for the BS-entropy.



12. A STRENGTHENED DPI FOR THE BS-ENTROPY

In this chapter, we provide a strengthening of the data processing inequality for the relative entropy introduced by Belavkin and Staszewski (BS-entropy). More specifically, we give analogous results to the ones in [CV17; CV18] for maximal f -divergences. For these, preservation of the maximal f -divergence is not equivalent to σ, ρ being recoverable in the sense of Petz, although the latter implies the former. Equivalent conditions to the preservation of a maximal f -divergence for the case in which Φ is a completely positive trace-preserving map are given in [HM17, Theorem 3.34].

In the current chapter, we provide two new equivalent conditions for the equality case of the data processing inequality for the BS-entropy and use them to obtain a strengthening of this inequality. Subsequently, we extend our result to a larger class of maximal f -divergences. Here, we first focus on quantum channels which are conditional expectations onto subalgebras and use the Stinespring dilation to lift our results to arbitrary quantum channels.

This chapter is structured as follows: Important results on standard and maximal f -divergences are reviewed in Section 12.1. In Section 12.2, we provide two new conditions which are equivalent to the preservation of the BS-entropy under a quantum channel. We use this result in Section 12.3 to prove our strengthened data processing inequality for the BS-entropy under a conditional expectation, which we subsequently generalize to other maximal f -divergences in Section 12.4. Finally, in Section 12.5, we extend this result to general quantum channels.

12.1 STANDARD AND MAXIMAL f -DIVERGENCES

12.1.1 STANDARD f -DIVERGENCES

In this subsection, we recall some definitions and basic properties concerning standard f -divergences. The main reference for them, as well as for maximal f -divergences is [HM17]. The latter are introduced in the next subsection. We focus on states with full rank and refer the reader to [HM17] for a more general treatment.

This is a picture of Munich, where part of the results presented in this chapter were obtained, and where I visited the Technische Universität München in January 2019.

Definition 12.1.1 — STANDARD f -DIVERGENCE, (HM17).

Let $f : (0, \infty) \rightarrow \mathbb{R}$ be an operator convex function and $\sigma > 0, \rho > 0$ be two unnormalized states on a matrix algebra \mathcal{M} . Then,

$$S_f(\sigma \| \rho) = \text{tr} \left[\rho^{1/2} f(L_\sigma R_{\rho^{-1}}) \rho^{1/2} \right]$$

is the *standard f -divergence*. This definition can be extended to general states σ, ρ as

$$S_f(\sigma \| \rho) := \lim_{\varepsilon \searrow 0} S_f(\sigma + \varepsilon I \| \rho + \varepsilon I).$$

Let us recall that, given $f : (0, \infty) \rightarrow \mathbb{R}$ an operator convex function, its *transpose* is given by

$$\tilde{f}(x) := xf(1/x).$$

We obtain the same standard f -divergence if we exchange ρ and σ and consider the transpose of f instead.

Proposition 12.1.2 — (HM17).

Let $f : (0, \infty) \rightarrow \mathbb{R}$ be an operator convex function with transpose \tilde{f} and $\sigma > 0, \rho > 0$ be two states on a matrix algebra \mathcal{M} . Then, $S_f(\sigma \| \rho) = S_{\tilde{f}}(\rho \| \sigma)$.

As we can see below, the main examples of standard f -divergences are directly connected to the well-known Umegaki relative entropy and standard Rényi divergences.

■ Example 12.1.3 — (HM17).

Let $f(x) = s(\alpha)x^\alpha$ for some $\alpha \in (0, \infty)$, where $s(\alpha) := -1$ for $0 < \alpha < 1$ and $s(\alpha) := 1$ for $\alpha \geq 1$. Then,

$$S_f(\sigma \| \rho) = s(\alpha) \text{tr} [\sigma^\alpha \rho^{1-\alpha}].$$

These quantities can be used to define the standard Rényi divergences. ■

■ Example 12.1.4 — (HM17).

Let $f(x) = x \log x$. Then,

$$S_f(\sigma \| \rho) = \text{tr} [\sigma (\log \sigma - \log \rho)]$$

defines the standard (Umegaki) relative entropy, usually denoted by $D(\sigma \| \rho)$. ■

Standard f -divergences extend the usual quantum relative entropy in more than one sense, since they share many of the properties that characterize the former, such as continuity (with respect to the first variable) or joint convexity. Indeed, one of the main features of this family of quantities is the data processing inequality.

Proposition 12.1.5 — DATA PROCESSING, (HM17).

Let $\Phi : \mathcal{M} \rightarrow \mathcal{B}$ be a trace-preserving map between matrix algebras \mathcal{M} and \mathcal{B} such that its dual map is a 2-positive trace-preserving map. Then, for every two states $\sigma > 0, \rho > 0$ on \mathcal{M} and every operator convex function $f : (0, \infty) \rightarrow \mathbb{R}$,

$$S_f(\Phi(\sigma) \| \Phi(\rho)) \leq S_f(\sigma \| \rho). \quad (12.1)$$

The above proposition in particular holds for quantum channels. Let us now recall the definition of the following map [HM17, Equation (3.19)] for Φ as in Proposition 12.1.5:

$$\mathcal{R}_\Phi^\rho(X) := \rho^{1/2} \Phi^* \left(\Phi(\rho)^{-1/2} (X) \Phi(\rho)^{-1/2} \right) \rho^{1/2} \quad \forall X \in \mathcal{B}.$$

This is the Petz recovery map for Φ with respect to ρ . In the following, we will assume that Φ preserves invertibility, as this will be the case in the situations addressed in this chapter.

A natural question is to ask for conditions for when the data processing inequality (12.1) holds with equality. Theorem 3.18 of [HM17] gives a list of equivalent conditions, from which we only state some:

Theorem 12.1.6 — (HM17).

Let $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra \mathcal{M} and let $\Phi : \mathcal{M} \rightarrow \mathcal{B}$ be a 2-positive trace-preserving linear map, where \mathcal{B} is again a matrix algebra. Then, the following are equivalent:

1. There exists a trace-preserving positive map $\Psi : \mathcal{B} \rightarrow \mathcal{M}$ such that $\Psi(\Phi(\rho)) = \rho$ and $\Psi(\Phi(\sigma)) = \sigma$.
2. $S_f(\Phi(\sigma) \parallel \Phi(\rho)) = S_f(\sigma \parallel \rho)$ for some operator convex function on $(0, \infty)$ such that $f(0^+) < \infty$ and

$$|\text{supp } \mu_f| \geq |\text{spec}(L_\sigma R_{\rho^{-1}}) \cup \text{spec}(L_{\Phi(\sigma)} R_{\Phi(\rho)^{-1}})|,$$

with μ_f the measure appearing in [Hia+11, Theorem 8.1].

3. $S_f(\Phi(\sigma) \parallel \Phi(\rho)) = S_f(\sigma \parallel \rho)$ for all operator convex f on $[0, \infty)$.
4. $\mathcal{R}_\Phi^\rho(\Phi(\sigma)) = \sigma$.

In particular, point (1) of Theorem 12.1.6 is symmetric in σ and ρ such that we obtain the following result, which was previously proven by Petz [Pet03].

Corollary 12.1.7 — (Pet03).

Let $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra \mathcal{M} and let $\Phi : \mathcal{M} \rightarrow \mathcal{B}$ be a 2-positive trace-preserving linear map, where \mathcal{B} is a matrix algebra. Then,

$$D(\sigma \parallel \rho) = D(\Phi(\sigma) \parallel \Phi(\rho)) \Leftrightarrow \sigma = \mathcal{R}_\Phi^\rho \circ \Phi(\sigma).$$

Moreover,

$$\sigma = \mathcal{R}_\Phi^\rho \circ \Phi(\sigma) \Leftrightarrow \rho = \mathcal{R}_\Phi^\sigma \circ \Phi(\rho).$$

12.1.2 MAXIMAL f -DIVERGENCES

In this subsection, we introduce maximal f -divergences and present some of their most basic properties. We also compare them to the aforementioned standard f -divergences. Again, we focus on states with full rank and refer the reader to [HM17] for the general case.

Definition 12.1.8 — MAXIMAL f -DIVERGENCE, (HM17).

Let $f : (0, \infty) \rightarrow \mathbb{R}$ be an operator convex function and $\sigma > 0$, $\rho > 0$ be two unnormalized states on a matrix algebra \mathcal{M} . Then,

$$\hat{S}_f(\sigma \parallel \rho) = \text{tr} \left[\rho^{1/2} f(\rho^{-1/2} \sigma \rho^{-1/2}) \rho^{1/2} \right]$$

is the *maximal f -divergence*. This definition can be extended to not necessarily full-rank states σ, ρ as

$$\hat{S}_f(\sigma \parallel \rho) := \lim_{\varepsilon \searrow 0} \hat{S}_f(\sigma + \varepsilon I \parallel \rho + \varepsilon I).$$

Again, the maximal f -divergences are identical if we exchange the states and replace f by its transpose.

Proposition 12.1.9 — (HM17).

Let $f : (0, \infty) \rightarrow \mathbb{R}$ be an operator convex function with transpose \tilde{f} and $\sigma > 0, \rho > 0$ be two states on a matrix algebra \mathcal{M} . Then, $\hat{S}_f(\sigma \| \rho) = \hat{S}_{\tilde{f}}(\rho \| \sigma)$.

The main example of a maximal f -divergence is the so-called BS-entropy, introduced by Belavkin and Staszewski in [BS82].

■ Example 12.1.10 — (HM17).

Let $f(x) = x \log x$. Then,

$$\hat{S}_f(\sigma \| \rho) = \text{tr} \left[\rho^{1/2} \sigma \rho^{-1/2} \log \left(\rho^{-1/2} \sigma \rho^{-1/2} \right) \right] = \text{tr} \left[\sigma \log \left(\sigma^{1/2} \rho^{-1} \sigma^{1/2} \right) \right]$$

is the *Belavkin-Staszewski relative entropy (BS-entropy)*. ■

Throughout this manuscript, we will use $\hat{S}_{BS}(\cdot \| \cdot)$ to denote the BS-entropy. However, it is common to find in the literature the notation $D_{BS}(\cdot \| \cdot)$ for this quantity.

Remarkably, this family of f -divergences also satisfies a data processing inequality, which makes them interesting quantities for information processing.

Proposition 12.1.11 — DATA PROCESSING, (HM17).

Let $\sigma > 0, \rho > 0$ be two states on a matrix algebra \mathcal{M} and $\Phi : \mathcal{M} \rightarrow \mathcal{B}$ be a trace-preserving positive linear map, where \mathcal{B} is a matrix algebra. Then,

$$\hat{S}_f(\Phi(\sigma) \| \Phi(\rho)) \leq \hat{S}_f(\sigma \| \rho).$$

As in the case of standard f -divergences, a natural question that arises is to characterize the states for which equality is fulfilled in the previous inequality. Some equivalent conditions for equality are collected in the following result, extracted from the larger list that appears in Theorem 3.34 of [HM17].

Theorem 12.1.12 — (HM17).

Let $\sigma > 0, \rho > 0$ be two states on a matrix algebra \mathcal{M} and $\Phi : \mathcal{M} \rightarrow \mathcal{B}$ be a trace-preserving positive linear map, where \mathcal{B} is a matrix algebra. Then the following are equivalent:

1. $\hat{S}_f(\Phi(\sigma) \| \Phi(\rho)) = \hat{S}_f(\sigma \| \rho)$ for some non-linear operator convex function f on $[0, \infty)$.
2. $\hat{S}_f(\Phi(\sigma) \| \Phi(\rho)) = \hat{S}_f(\sigma \| \rho)$ for all operator convex functions f on $[0, \infty)$.
3. $\text{tr}[\Phi(\sigma)^2 \Phi(\rho)^{-1}] = \text{tr}[\sigma^2 \rho^{-1}]$.

R Remark 12.1.13

The function in point (3) of the above theorem is $S_f(\sigma \| \rho) = \hat{S}_f(\sigma \| \rho)$ for $f(x) = x^2$. Indeed, it is true that if f is a polynomial of degree at most 2, the maximal and the standard f -divergences coincide.

Another natural question that arises is whether the conditions listed above are equivalent to those of equality in the data processing inequality for standard f -divergences that appeared in Theorem 12.1.6. We will later see that this is not the case in general. The following result shows how standard and maximal f -divergences are related for the same operator convex function f .

Proposition 12.1.14 — (HM17).

For every two states $\sigma > 0$, $\rho > 0$ on a matrix algebra \mathcal{M} and every operator convex function $f : (0, \infty) \rightarrow \mathbb{R}$,

$$S_f(\sigma \parallel \rho) \leq \hat{S}_f(\sigma \parallel \rho). \quad (12.2)$$

R Remark 12.1.15

When σ and ρ commute, given an operator convex function f the maximal f -divergence coincides with the standard f -divergence, and both of them coincide with the classical ones introduced in [AS66] [Csi67]. In fact, the inequality (12.2) is strict for states which do not commute, provided f is “complicated enough” [HM17]. For qubits, this is the case for any f which is not a polynomial [HM17].

R Remark 12.1.16

Recoverability easily implies $\hat{S}_f(\Phi(\sigma) \parallel \Phi(\rho)) = \hat{S}_f(\sigma \parallel \rho)$. The fact that the left hand side is smaller than or equal to the right hand side follows from the data processing inequality. For the other inequality, it is enough to consider the case $f(x) = x^2$. Then, $\hat{S}_f(\sigma \parallel \rho) = \text{tr}[\sigma^2 \rho^{-1}]$. By assumption,

$$\sigma = \rho^{1/2} \Phi^*(\Phi(\rho)^{-1/2} \Phi(\sigma) \Phi(\rho)^{-1/2}) \rho^{1/2}$$

and

$$\begin{aligned} \text{tr}[\sigma^2 \rho^{-1}] &= \text{tr}[\rho(\Phi^*(\Phi(\rho)^{-1/2} \Phi(\sigma) \Phi(\rho)^{-1/2}))^2] \\ &\leq \text{tr}[\Phi(\rho)(\Phi(\rho)^{-1/2} \Phi(\sigma) \Phi(\rho)^{-1/2})^2] \\ &= \text{tr}[\Phi(\sigma)^2 \Phi(\rho)^{-1}] \end{aligned}$$

The second line is from Jensen’s operator inequality (Theorem 4.4.6).

R Remark 12.1.17

In general, preservation of maximal f -divergences does not imply recoverability by means of the Petz recovery map. However, for unital qubit channels, it does [HM17]. This does not contradict Remark 12.1.2, since Φ can still preserve both maximal and standard f -divergences, even if their value is not the same.

12.2 A CONDITION FOR EQUALITY

Theorem 3.34 of [HM17] lists several equivalent conditions for the preservation of maximal f -divergences under a quantum channel. We will prove two other equivalent conditions, inspired by [Pet03]. We need the following technical proposition in the proof of the main result.

Proposition 12.2.1 — (BC19b).

Let \mathcal{M} be two matrix algebras. We consider two quantum states $\sigma > 0$ and $\rho > 0$ on \mathcal{M} and a completely positive trace-preserving map $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ such that $\sigma_{\mathcal{T}}, \rho_{\mathcal{T}} > 0$. Let $U : \mathcal{N} \rightarrow \mathcal{M}$ be given by $U(X) = \sigma^{1/2} \mathcal{T}^*(\sigma_{\mathcal{T}}^{-1/2} X)$ for all $X \in \mathcal{N}$. Then, $U^*(Y) = \sigma_{\mathcal{T}}^{-1/2} \mathcal{T}(\sigma^{1/2} Y)$ for every $Y \in \mathcal{M}$ and

$$U^* \Gamma U \leq \Gamma_{\mathcal{T}},$$

Moreover, $U^* U \leq \text{Id}$. If \mathcal{N} is a unital subalgebra of \mathcal{M} and $\mathcal{T} = \mathcal{E}$, where \mathcal{E} is the conditional expectation onto \mathcal{N} , we can extend U to an operator on \mathcal{M} and it holds that $U^* U = \mathcal{E}$.

Proof. The form of U^* follows from direct computation. Let $X \in \mathcal{N}$. Then,

$$\begin{aligned}
 \langle X, U^* \Gamma U(X) \rangle &= \langle U(X), \Gamma U(X) \rangle \\
 &= \langle \sigma^{1/2} \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1/2} X), \sigma^{-1/2} \rho \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1/2} X) \rangle \\
 &= \text{tr} \left[\rho \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1/2} X) \mathcal{T}^* (X^* \sigma_{\mathcal{T}}^{-1/2}) \right] \\
 &\leq \text{tr} \left[\rho \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1/2} X X^* \sigma_{\mathcal{T}}^{-1/2}) \right] \\
 &= \langle X, \Gamma_{\mathcal{T}} X \rangle.
 \end{aligned}$$

The fourth line follows by the Schwarz inequality. Hence, $U^* \Gamma U \leq \Gamma_{\mathcal{N}}$. A similar calculation yields

$$\begin{aligned}
 \langle X, U^* U(X) \rangle &= \langle U(X), U(X) \rangle \\
 &= \langle \sigma^{1/2} \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1/2} X), \sigma^{1/2} \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1/2} X) \rangle \\
 &= \text{tr} \left[\sigma \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1/2} X) \mathcal{T}^* (X^* \sigma_{\mathcal{T}}^{-1/2}) \right] \\
 &\leq \text{tr} \left[\sigma \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1/2} X X^* \sigma_{\mathcal{T}}^{-1/2}) \right] \\
 &= \langle X, X \rangle.
 \end{aligned}$$

This implies $U^* U \leq \text{Id}$. In the case where \mathcal{T} is a conditional expectation, we can write $U(X) = \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \mathcal{E}(X)$ for all $X \in \mathcal{M}$. The Equation $U^* U = \mathcal{E}$ then follows from a similar calculation to the one above and the fact that \mathcal{E} is trace preserving. ■

Now we can state and prove the new equivalent condition for equality between BS-entropies under the application of a quantum channel.

Theorem 12.2.2 — A CONDITION FOR EQUALITY IN THE DPI FOR THE BS ENTROPY, (BC19b).

Let \mathcal{M}, \mathcal{N} be two matrix algebras and $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ be a completely positive trace-preserving map. Let $\sigma > 0, \rho > 0$ be two quantum states on \mathcal{M} such that $\mathcal{T}(\sigma) > 0, \mathcal{T}(\rho) > 0$. Then

$$\hat{S}_{\text{BS}}(\sigma \| \rho) = \hat{S}_{\text{BS}}(\sigma_{\mathcal{T}} \| \rho_{\mathcal{T}}) \quad (12.3)$$

if and only if

$$\mathcal{T}^* (\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}}) = \sigma^{-1} \rho. \quad (12.4)$$

Proof. The proof follows the proof of [Pet03, Theorem 3.1]. Let $U : \mathcal{N} \rightarrow \mathcal{M}$ be defined as $U(X) = \sigma^{1/2} \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1/2} X)$ for all $X \in \mathcal{N}$. Using the integral representation of the operator monotone function $\log(x)$,

$$\log x = \int_0^\infty \left(\frac{1}{1+t} - \frac{1}{t+x} \right) dt,$$

we infer below that Equation (12.3) is equivalent to

$$\left\langle \sigma_{\mathcal{T}}^{1/2}, U^* ((\Gamma + t)^{-1} - (t+1)^{-1} I) U \sigma_{\mathcal{T}}^{1/2} \right\rangle = \left\langle \sigma_{\mathcal{T}}^{1/2}, ((\Gamma_{\mathcal{T}} + t)^{-1} - (t+1)^{-1} I) \sigma_{\mathcal{T}}^{1/2} \right\rangle. \quad (12.5)$$

Indeed, we know that $\Gamma_{\mathcal{T}} \geq U^* \Gamma U$ and $U^* U \leq \text{Id}$ (see Proposition 12.2.1). Let $f_t(x) = (t+x)^{-1} - t^{-1}$ for fixed $t \geq 0$. Since $x \mapsto x^{-1}$ is operator monotone decreasing and operator convex on $(0, \infty)$, the same property holds for $f_t(x)$ on $[0, \infty)$ for $t > 0$. Hence,

$$(U^* \Gamma U + t)^{-1} - t^{-1} I \geq (\Gamma_{\mathcal{T}} + t)^{-1} - t^{-1} I.$$

Moreover, $f_t(x) \leq 0$ for every $x \geq 0$. Using [Bha97, Theorem V.2.3] and the fact that U is a contraction, it holds that

$$U^* ((\Gamma + t)^{-1} - t^{-1}I) U \geq (U^* \Gamma U + t)^{-1} - t^{-1}I,$$

and thus,

$$U^* ((\Gamma + t)^{-1} - t^{-1}I) U \geq (\Gamma_{\mathcal{T}} + t)^{-1} - t^{-1}I. \quad (12.6)$$

Hence, since $U(\sigma_{\mathcal{T}}^{1/2}) = \sigma^{1/2}$,

$$\begin{aligned} \hat{S}_{\text{BS}}(\sigma \| \rho) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{T}} \| \rho_{\mathcal{T}}) &= \int_0^\infty \left\langle \sigma^{1/2}, ((\Gamma + t)^{-1} - (t+1)^{-1}I) \sigma^{1/2} \right\rangle dt \\ &\quad - \int_0^\infty \left\langle \sigma_{\mathcal{T}}^{1/2}, ((\Gamma_{\mathcal{T}} + t)^{-1} - (t+1)^{-1}I) \sigma_{\mathcal{T}}^{1/2} \right\rangle dt \\ &= \int_0^\infty \left\langle \sigma_{\mathcal{T}}^{1/2}, (U^* (\Gamma + t)^{-1} U - (\Gamma_{\mathcal{T}} + t)^{-1}) \sigma_{\mathcal{T}}^{1/2} \right\rangle dt \\ &\geq 0, \end{aligned}$$

where the last inequality follows from Equation (12.6). Moreover, since for every $t > 0$ the infinitesimal term at time t inside the integral is always non-negative, the difference of BS-entropies vanishes if and only if every infinitesimal term does. Therefore, Equation (12.3) is equivalent to Equation (12.5), and they both imply

$$U^* (\Gamma + t)^{-1} \sigma^{1/2} = (\Gamma_{\mathcal{T}} + t)^{-1} \sigma_{\mathcal{T}}^{1/2}$$

for all $t > 0$. Differentiating with respect to t gives

$$U^* (\Gamma + t)^{-2} \sigma^{1/2} = (\Gamma_{\mathcal{T}} + t)^{-2} \sigma_{\mathcal{T}}^{1/2}.$$

It follows that

$$\begin{aligned} \left\| U^* (\Gamma + t)^{-1} \sigma^{1/2} \right\|_2^2 &= \left\langle \sigma_{\mathcal{T}}^{1/2}, (\Gamma_{\mathcal{T}} + t)^{-2} \sigma_{\mathcal{T}}^{1/2} \right\rangle \\ &= \left\langle \sigma_{\mathcal{T}}^{1/2}, U^* (\Gamma + t)^{-2} \sigma^{1/2} \right\rangle \\ &= \left\| (\Gamma + t)^{-1} \sigma^{1/2} \right\|_2^2. \end{aligned}$$

We have shown $\langle A, UU^* A \rangle = \langle A, A \rangle$ for some $A \in \mathcal{M}$ and we know $UU^* \leq \text{Id}$ since $\|U\|_\infty = \|U^*\|_\infty$, thus we infer $UU^* A = A$. Therefore, we have arrived at

$$U(\Gamma_{\mathcal{T}} + t)^{-1} \sigma_{\mathcal{T}}^{1/2} = UU^* (\Gamma + t)^{-1} \sigma^{1/2} = (\Gamma + t)^{-1} \sigma^{1/2}$$

Differentiating again with respect to t , it follows that

$$U(\Gamma_{\mathcal{T}} + t)^{-n} \sigma_{\mathcal{T}}^{1/2} = (\Gamma + t)^{-n} \sigma^{1/2}$$

for all $n \in \mathbb{N}$ and hence also

$$Uf(\Gamma_{\mathcal{T}}) \sigma_{\mathcal{T}}^{1/2} = f(\Gamma) \sigma^{1/2}$$

for all continuous functions f by the Stone-Weierstrass theorem. For $f(x) = x$, we obtain

$$\sigma^{1/2} \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}}) = \sigma^{-1/2} \rho.$$

This proves the first implication. The reverse implication follows from

$$\mathrm{tr}[\rho_{\mathcal{T}}^2 \sigma_{\mathcal{T}}^{-1}] = \mathrm{tr}[\rho \mathcal{T}^*(\rho_{\mathcal{T}} \sigma_{\mathcal{T}}^{-1})] = \mathrm{tr}[\rho^2 \sigma^{-1}]$$

and the fact that

$$\mathrm{tr}[\rho_{\mathcal{T}}^2 \sigma_{\mathcal{T}}^{-1}] = \mathrm{tr}[\rho^2 \sigma^{-1}] \Leftrightarrow \mathrm{tr}[\sigma_{\mathcal{T}}^2 \rho_{\mathcal{T}}^{-1}] = \mathrm{tr}[\sigma^2 \rho^{-1}]$$

by Theorem 12.1.12 for $f(x) = x^{1/2}$, $\tilde{f} = f(x)$. ■

R Remark 12.2.3

Note that Equation (12.4) can be rephrased as a recovery condition for ρ from σ under the application of a quantum channel:

$$\rho = \sigma \mathcal{T}^*(\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}}),$$

as well as exchanging the roles of ρ and σ .

R Remark 12.2.4

In the particular case in which the map is a trace-preserving conditional expectation \mathcal{E} onto a unital matrix subalgebra \mathcal{N} of \mathcal{M} , Theorem 12.2.2 can be written as follows:

$$\hat{S}_{\mathrm{BS}}(\sigma || \rho) = \hat{S}_{\mathrm{BS}}(\sigma_{\mathcal{N}} || \rho_{\mathcal{N}})$$

if and only if

$$\sigma_{\mathcal{N}}^{-1} \rho_{\mathcal{N}} = \sigma^{-1} \rho.$$

Here, we have assumed $\sigma > 0$, $\rho > 0$. In this case, the recovery condition for ρ from σ under the application of a conditional expectation is stated as follows:

$$\rho = \sigma \sigma_{\mathcal{N}}^{-1} \rho_{\mathcal{N}}.$$

We can further see that, for quantum channels, the condition appearing in Equation (12.3) is implied by another one involving Γ and $\Gamma_{\mathcal{T}}$ which will appear in the main result of Section 12.5.

Proposition 12.2.5 — (BC19b).

Let \mathcal{M} be a matrix algebra and let $\sigma > 0$, $\rho > 0$ be two states on it. Let \mathcal{N} be another matrix algebra and let $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ be a quantum channel. Let V be the isometry associated to a Stinespring dilation (Theorem 4.4.9) of \mathcal{T} . If the following expression holds

$$V \sigma^{1/2} V^* \left(\sigma_{\mathcal{T}}^{-1/2} \Gamma_{\mathcal{T}}^{1/2} \sigma_{\mathcal{T}}^{1/2} \otimes I \right) = V \Gamma^{1/2} \sigma^{1/2} V^*, \quad (12.7)$$

then

$$\sigma^{-1} \rho = \mathcal{T}^*(\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}}).$$

Proof. Using Equation (12.7), and abbreviating $\Theta := \sigma_{\mathcal{T}}^{-1/2} \Gamma_{\mathcal{T}}^{1/2} \sigma_{\mathcal{T}}^{1/2} \otimes I$, we can see that

$$\begin{aligned} V \Gamma \sigma^{1/2} V^* &= V \Gamma^{1/2} V^* V \Gamma^{1/2} \sigma^{1/2} V^* \\ &= V \Gamma^{1/2} V^* V \sigma^{1/2} V^* \Theta \\ &= V \Gamma^{1/2} \sigma^{1/2} V^* \Theta \\ &= V \sigma^{1/2} V^* \Theta^2. \end{aligned}$$

Now, note that

$$\Theta^2 = \sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}} \otimes I.$$

Hence, multiplying the expression above by $V^*(\cdot)V$ and using $\mathcal{T}^*(X) = V^*(X \otimes I)V$ for all $X \in \mathcal{N}$, we get

$$\begin{aligned} \Gamma \sigma^{1/2} &= \sigma^{1/2} V^* (\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}} \otimes I) V \\ &= \sigma^{1/2} \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}}), \end{aligned}$$

which is equivalent to

$$\sigma^{-1} \rho = \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}}).$$

■

R Remark 12.2.6

The converse implication is also true, although we cannot prove it directly here. However, it can be obtained as a consequence of Theorem 12.5.1. Note also that multiplying directly Equation (12.7) by $V^*(\cdot)V$, we get the following expression:

$$\sigma^{1/2} V^* (\sigma_{\mathcal{T}}^{-1/2} \Gamma_{\mathcal{T}} \sigma_{\mathcal{T}}^{1/2} \otimes I) V = \Gamma^{1/2} \sigma^{1/2},$$

which can be rewritten as

$$\sigma^{1/2} \mathcal{T}^* (\sigma_{\mathcal{T}}^{-1/2} \Gamma_{\mathcal{T}}^{1/2} \sigma_{\mathcal{T}}^{1/2}) = \Gamma^{1/2} \sigma^{1/2}. \quad (12.8)$$

For conditional expectations, this condition can be actually seen to be equivalent to Equation (12.3).

Proposition 12.2.7 — (BC19b).

Let \mathcal{M} be a matrix algebra, \mathcal{N} be a unital matrix subalgebra, and $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ be the unique trace-preserving conditional expectation onto \mathcal{N} . Let $\sigma > 0$, $\rho > 0$ and define $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$, $\rho_{\mathcal{N}} := \mathcal{E}(\rho)$. Then,

$$\rho = \sigma \sigma_{\mathcal{N}}^{-1} \rho_{\mathcal{N}} \quad (12.9)$$

is equivalent to

$$\sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} \sigma_{\mathcal{N}}^{1/2} = \Gamma^{1/2} \sigma^{1/2}. \quad (12.10)$$

Proof. Recalling the explicit expressions for Γ and $\Gamma_{\mathcal{N}}$, Equation (12.9) can be seen to be equivalent to

$$\sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}} = \Gamma \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2},$$

and iterating n times, we get

$$\sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^n = \Gamma^n \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2}.$$

By the Weierstrass theorem, this implies

$$\sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} f(\Gamma_{\mathcal{N}}) = f(\Gamma) \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2},$$

for every continuous function f , and, in particular, for $f(x) = x^{1/2}$, we have

$$\sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} = \Gamma^{1/2} \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2}. \quad (12.11)$$

This concludes (12.9) \implies (12.10). The converse implication follows from Equation (12.11), iterating it twice. ■

Equation (12.10) will appear in the main result of Section 12.3. As a direct consequence of Theorem 12.2.2 for conditional expectations and Proposition 12.2.7, we have the following result.

Corollary 12.2.8 — (BC19b).

Under the conditions of the proposition above, the following facts are equivalent:

1. $\hat{S}_{BS}(\sigma\|\rho) = \hat{S}_{BS}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}})$.
2. $\rho = \sigma\sigma_{\mathcal{N}}^{-1}\rho_{\mathcal{N}}$.
3. $\sigma^{1/2}\sigma_{\mathcal{N}}^{-1/2}\Gamma_{\mathcal{N}}^{1/2}\sigma_{\mathcal{N}}^{1/2} = \Gamma^{1/2}\sigma^{1/2}$.

Let us denote the aforementioned asymmetric recovery map, which we will call *BS recovery condition* throughout the rest of the chapter, by

$$\mathcal{B}_{\mathcal{T}}^{\sigma}(\cdot) := \sigma\mathcal{T}^*(\sigma_{\mathcal{T}}^{-1}(\cdot)).$$

Note that, although $\mathcal{B}_{\mathcal{T}}^{\sigma}$ is trace-preserving, it is not completely positive in general. Moreover, analogously to Theorem 12.1.7, Theorem 12.2.2 can be restated as

$$\hat{S}_{BS}(\sigma\|\rho) = \hat{S}_{BS}(\sigma_{\mathcal{T}}\|\rho_{\mathcal{T}}) \Leftrightarrow \rho = \mathcal{B}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho). \quad (12.12)$$

R Remark 12.2.9

Note that, analogously to the case for the relative entropy, from Remark 12.2 and Equation (12.12) we can deduce

$$\begin{aligned} \hat{S}_{BS}(\sigma\|\rho) = \hat{S}_{BS}(\sigma_{\mathcal{T}}\|\rho_{\mathcal{T}}) &\Leftrightarrow \rho = \mathcal{B}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho) \\ &\Leftrightarrow \sigma = \mathcal{B}_{\mathcal{T}}^{\rho} \circ \mathcal{T}(\sigma) \\ &\Leftrightarrow \hat{S}_{BS}(\rho\|\sigma) = \hat{S}_{BS}(\rho_{\mathcal{T}}\|\sigma_{\mathcal{T}}). \end{aligned}$$

Here, the second equivalence follows from Theorem 12.1.12 and the fact that $\tilde{f}(x) = f(x)$ for $f(x) = x^{1/2}$.

Now, a natural question is whether σ can be recovered in the sense of Petz in the same cases that it can be recovered in the sense of the BS-entropy, and thus, whether the conditions of equality for the relative entropy coincide with those of equality for the BS-entropy. This can be answered negatively in general, although one implication always holds.

Indeed, from [Pet03, Theorem 2], we can see that $D(\sigma\|\rho) = D(\sigma_{\mathcal{T}}\|\rho_{\mathcal{T}})$ is equivalent to

$$\mathcal{T}^*(\sigma_{\mathcal{T}}^{it}\rho_{\mathcal{T}}^{-it}) = \sigma^{it}\rho^{-it} \text{ for every } t \in \mathbb{R},$$

and by analytic continuation, it implies

$$\mathcal{T}^*(\sigma_{\mathcal{T}}^z\rho_{\mathcal{T}}^{-z}) = \sigma^z\rho^{-z} \text{ for every } z \in \mathbb{C}.$$

In particular,

$$D(\sigma\|\rho) = D(\sigma_{\mathcal{T}}\|\rho_{\mathcal{T}}) \implies \mathcal{T}^*(\sigma_{\mathcal{T}}^{-1}\rho_{\mathcal{T}}) = \sigma^{-1}\rho,$$

obtaining the following well-known result:

Corollary 12.2.10 Let $\sigma, \rho > 0$ be states on \mathcal{M} and such that $\sigma_{\mathcal{T}}, \rho_{\mathcal{T}} > 0$ for $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ a quantum channel. Then,

$$D(\sigma \| \rho) = D(\sigma_{\mathcal{T}} \| \rho_{\mathcal{T}}) \implies \hat{S}_{BS}(\sigma \| \rho) = \hat{S}_{BS}(\sigma_{\mathcal{T}} \| \rho_{\mathcal{T}}).$$

Equivalently,

$$\sigma = \mathcal{R}_{\mathcal{T}}^{\rho} \circ \mathcal{T}(\sigma) \implies \sigma = \mathcal{B}_{\mathcal{T}}^{\rho} \circ \mathcal{T}(\sigma).$$

The converse implications are false in general. Indeed, [JPP09, Example 2.2] and [HM17, Example 4.8] constitute examples of states for which there is equality between BS-entropies but one state cannot be recovered from the other using the Petz recovery map.

12.3 STRENGTHENED DATA PROCESSING INEQUALITY FOR THE BS-ENTROPY

The well-known data processing inequality for the partial trace, whose extension for standard f -divergences is Proposition 12.1.5, finds its analogue for maximal f -divergences in Proposition 12.1.11. In the main result of this section, inspired by [CV17], we will prove a strengthened lower bound for the data processing inequality for the BS-entropy when the map considered is a trace-preserving conditional expectation onto a unital matrix subalgebra \mathcal{N} of \mathcal{M} . We will present an extension of this result to general quantum channels in Section 12.5. Before we start, we introduce some important tools.

Lemma 12.3.1 — (BC19b).

Let \mathcal{M} be a matrix algebra with unital subalgebra \mathcal{N} . Let $\sigma > 0, \rho > 0$ be two quantum states on \mathcal{M} and consider $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ the unique trace-preserving conditional expectation onto this subalgebra. Consider $U : \mathcal{M} \rightarrow \mathcal{M}$ defined as in Proposition 12.2.1. Then

$$\left\langle \sigma_{\mathcal{N}}^{1/2}, (U^*(\Gamma + t)^{-1}U - (\Gamma_{\mathcal{N}} + t)^{-1}) \sigma_{\mathcal{N}}^{1/2} \right\rangle \geq t \left\| (U(\Gamma_{\mathcal{N}} + t)^{-1} - (\Gamma + t)^{-1}U) \sigma_{\mathcal{N}}^{1/2} \right\|_2^2,$$

for every $t > 0$.

Proof. By virtue of [CV17, Lemma 2.1], we know that

$$\left\langle \sigma_{\mathcal{N}}^{1/2}, U^*(\Gamma + t)^{-1}U \sigma_{\mathcal{N}}^{1/2} \right\rangle = \left\langle \sigma_{\mathcal{N}}^{1/2}, (\Gamma_{\mathcal{N}} + t)^{-1} \sigma_{\mathcal{N}}^{1/2} \right\rangle + \langle w_t, (\Gamma + t)w_t \rangle,$$

for

$$w_t := U(\Gamma_{\mathcal{N}} + t)^{-1} \sigma_{\mathcal{N}}^{1/2} - (\Gamma + t)^{-1}U \sigma_{\mathcal{N}}^{1/2}.$$

Hence, taking into account that

$$\langle w_t, (\Gamma + t)w_t \rangle \geq t \|w_t\|_2^2,$$

we get

$$\left\langle \sigma_{\mathcal{N}}^{1/2}, (U^*(\Gamma + t)^{-1}U - (\Gamma_{\mathcal{N}} + t)^{-1}) \sigma_{\mathcal{N}}^{1/2} \right\rangle \geq t \left\| (U(\Gamma_{\mathcal{N}} + t)^{-1} - (\Gamma + t)^{-1}U) \sigma_{\mathcal{N}}^{1/2} \right\|_2^2.$$

■

We need another tool before we can prove the main result of this section.

Proposition 12.3.2 — (BC19b).

Consider two quantum states $\rho, \sigma > 0$ on \mathcal{M} and their expectations $\rho_{\mathcal{N}}$ and $\sigma_{\mathcal{N}}$ on $\mathcal{N} \subset \mathcal{M}$. Define $\Gamma = \sigma^{-1/2} \rho \sigma^{-1/2}$ and $\Gamma_{\mathcal{N}} = \sigma_{\mathcal{N}}^{-1/2} \rho_{\mathcal{N}} \sigma_{\mathcal{N}}^{-1/2}$. Then,

$$\|\Gamma_{\mathcal{N}}\|_{\infty} \leq \|\Gamma\|_{\infty}.$$

Proof. Let us introduce the norm $\|A\|_{\infty, \mathcal{A}}$ for \mathcal{A} some unital subalgebra of $\mathcal{B}(\mathcal{H})$ and $A : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ a linear map. The norm is defined as

$$\|A\|_{\infty, \mathcal{A}} := \sup_{B \in \mathcal{A}} \frac{\|A(B)\|_2}{\|B\|_2}.$$

We note that \mathcal{N} and \mathcal{M} form a Hilbert space with the Hilbert-Schmidt norm and the bounded operators on this Hilbert space form a C^* -algebra with the above norms (for $\mathcal{A} = \mathcal{M}$ and \mathcal{N} , respectively). Furthermore,

$$\|\Gamma_{\mathcal{N}}\|_{\infty, \mathcal{M}} = \|\Gamma_{\mathcal{N}}\|_{\infty, \mathcal{N}} = \|\sigma_{\mathcal{N}}^{-1/2} \rho_{\mathcal{N}} \sigma_{\mathcal{N}}^{-1/2}\|_{\infty},$$

since

$$\|\Gamma_{\mathcal{N}}\|_{\infty, \mathcal{M}} \leq \sup_{B \in \mathcal{M}} \frac{\|\sigma_{\mathcal{N}}^{-1/2} \rho_{\mathcal{N}} \sigma_{\mathcal{N}}^{-1/2}\|_{\infty} \|B\|_2}{\|B\|_2}$$

and

$$\|\Gamma_{\mathcal{N}}\|_{\infty, \mathcal{N}} \geq \frac{\|\sigma_{\mathcal{N}}^{-1/2} \rho_{\mathcal{N}} \sigma_{\mathcal{N}}^{-1/2} P\|_2}{\|P\|_2} = \|\sigma_{\mathcal{N}}^{-1/2} \rho_{\mathcal{N}} \sigma_{\mathcal{N}}^{-1/2}\|_{\infty},$$

where P is the projection on the eigenspace of the largest eigenvalue of $\sigma_{\mathcal{N}}^{-1/2} \rho_{\mathcal{N}} \sigma_{\mathcal{N}}^{-1/2}$. As \mathcal{N} is a von Neumann algebra, it holds that $P \in \mathcal{N}$ (see [BR79, Section 2.4.2]). Proposition 12.2.1 shows that $\Gamma_{\mathcal{N}} = U^* \Gamma U$ on $(\mathcal{N}, \langle \cdot, \cdot \rangle_{\text{HS}})$. Thus,

$$\begin{aligned} \|\sigma_{\mathcal{N}}^{-1/2} \rho_{\mathcal{N}} \sigma_{\mathcal{N}}^{-1/2}\|_{\infty} &= \|\Gamma_{\mathcal{N}}\|_{\infty, \mathcal{N}} \\ &= \|U^* \Gamma U\|_{\infty, \mathcal{N}} \\ &\leq \|U\|_{\infty, \mathcal{M}}^2 \|\Gamma\|_{\infty, \mathcal{M}} \\ &\leq \|\Gamma\|_{\infty}. \end{aligned}$$

The last line follows, since $U^* U = \mathcal{E}$, $\mathcal{E} \leq \text{Id}$ and therefore $\|U(B)\|_2^2 \leq \langle B, \mathcal{E}(B) \rangle \leq \|B\|_2^2$ for all $B \in \mathcal{M}$. \blacksquare

The main result of this section reads as follows.

Theorem 12.3.3 — STRENGTHENED DPI FOR THE BS-ENTROPY, (BC19b).

Let \mathcal{M} be a matrix algebra with unital subalgebra \mathcal{N} . Let $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma > 0, \rho > 0$ be two quantum states on \mathcal{M} . Then

$$\hat{S}_{\text{BS}}(\sigma \| \rho) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}}) \geq \left(\frac{\pi}{4}\right)^4 \|\Gamma\|_{\infty}^{-2} \left\| \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} \sigma_{\mathcal{N}}^{1/2} - \Gamma^{1/2} \sigma^{1/2} \right\|_2^4. \quad (12.13)$$

Proof. The first part of the proof follows the first part of the one of Theorem 12.2.2. Consider $U : \mathcal{M} \rightarrow \mathcal{M}$ as defined in Proposition 12.2.1. Then, the following inequality holds as operators on $(\mathcal{N}, \langle \cdot, \cdot \rangle_{\text{HS}})$

$$U^* ((\Gamma + t)^{-1} - (t+1)^{-1} I) U \geq (\Gamma_{\mathcal{N}} + t)^{-1} - (t+1)^{-1} I.$$

Therefore,

$$\begin{aligned}\hat{S}_{\text{BS}}(\sigma\|\rho) &= \int_0^\infty \left\langle \sigma_{\mathcal{N}}^{1/2}, U^* ((\Gamma+t)^{-1} - (t+1)^{-1}I) U \sigma_{\mathcal{N}}^{1/2} \right\rangle dt \\ &\geq \int_0^\infty \left\langle \sigma_{\mathcal{N}}^{1/2}, ((\Gamma_{\mathcal{N}}+t)^{-1} - (t+1)^{-1}I) \sigma_{\mathcal{N}}^{1/2} \right\rangle dt \\ &= \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}).\end{aligned}$$

Consider the infinitesimal expressions in the previous integrals. Hence, given $0 < T < \infty$, following the proof of [CV17, Theorem 1.7] and by virtue of the Cauchy-Schwarz inequality

$$\begin{aligned}\hat{S}_{\text{BS}}(\sigma\|\rho) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) &\geq \int_0^T \left\langle \sigma_{\mathcal{N}}^{1/2}, (U^*(\Gamma+t)^{-1}U - (\Gamma_{\mathcal{N}}+t)^{-1}) \sigma_{\mathcal{N}}^{1/2} \right\rangle dt \\ &\geq \int_0^T t \left\| (U(\Gamma_{\mathcal{N}}+t)^{-1} - (\Gamma+t)^{-1}U) \sigma_{\mathcal{N}}^{1/2} \right\|_2^2 dt \\ &\geq \frac{1}{T} \left(\int_0^T t^{1/2} \left\| (U(\Gamma_{\mathcal{N}}+t)^{-1} - (\Gamma+t)^{-1}U) \sigma_{\mathcal{N}}^{1/2} \right\|_2 dt \right)^2.\end{aligned}$$

Here, we have used Lemma 12.3.1 in the second line. Let us study the expression appearing in the last integral. For that, recall the integral representation of the operator monotone square root function,

$$x^{1/2} = \frac{1}{\pi} \int_0^\infty t^{1/2} \left(\frac{1}{t} - \frac{1}{t+x} \right) dt,$$

which clearly yields

$$U\Gamma_{\mathcal{N}}^{1/2}\sigma_{\mathcal{N}}^{1/2} - \Gamma^{1/2}U\sigma_{\mathcal{N}}^{1/2} = \frac{1}{\pi} \int_0^\infty t^{1/2} ((\Gamma+t)^{-1}U - U(\Gamma_{\mathcal{N}}+t)^{-1}) \sigma_{\mathcal{N}}^{1/2} dt.$$

The left hand side can be simplified as

$$U\Gamma_{\mathcal{N}}^{1/2}\sigma_{\mathcal{N}}^{1/2} - \Gamma^{1/2}U\sigma_{\mathcal{N}}^{1/2} = \sigma^{1/2}\sigma_{\mathcal{N}}^{-1/2}\Gamma_{\mathcal{N}}^{1/2}\sigma_{\mathcal{N}}^{1/2} - \Gamma^{1/2}\sigma^{1/2},$$

and thus

$$\begin{aligned}\left\| \sigma^{1/2}\sigma_{\mathcal{N}}^{-1/2}\Gamma_{\mathcal{N}}^{1/2}\sigma_{\mathcal{N}}^{1/2} - \Gamma^{1/2}\sigma^{1/2} \right\|_2 &= \frac{1}{\pi} \left\| \int_0^\infty t^{1/2} (U(\Gamma_{\mathcal{N}}+t)^{-1} - (\Gamma+t)^{-1}U) \sigma_{\mathcal{N}}^{1/2} dt \right\|_2 \\ &\leq \frac{1}{\pi} \int_0^T t^{1/2} \left\| (U(\Gamma_{\mathcal{N}}+t)^{-1} - (\Gamma+t)^{-1}U) \sigma_{\mathcal{N}}^{1/2} \right\|_2 dt \\ &\quad + \frac{1}{\pi} \left\| \int_T^\infty t^{1/2} (U(\Gamma_{\mathcal{N}}+t)^{-1} - (\Gamma+t)^{-1}U) \sigma_{\mathcal{N}}^{1/2} dt \right\|_2\end{aligned}$$

for any $0 < T < \infty$. We present now an upper bound for the last term on the right hand side. As shown in the proof of [CV17, Theorem 1.7],

$$\begin{aligned}&\left\| \int_T^\infty t^{1/2} (U(\Gamma_{\mathcal{N}}+t)^{-1} - (\Gamma+t)^{-1}U) \sigma_{\mathcal{N}}^{1/2} dt \right\|_2 \\ &\leq \left\| \int_T^\infty t^{1/2} (U(\Gamma_{\mathcal{N}}+t)^{-1} - t^{-1}U) \sigma_{\mathcal{N}}^{1/2} dt \right\|_2 + \left\| \int_T^\infty t^{1/2} (Ut^{-1} - (\Gamma+t)^{-1}U) \sigma_{\mathcal{N}}^{1/2} dt \right\|_2\end{aligned}$$

Moreover, we have

$$\int_T^\infty t^{1/2} (t^{-1}I - (\Gamma_{\mathcal{N}}+t)^{-1}) dt \leq \frac{2\|\Gamma_{\mathcal{N}}\|_\infty}{T^{1/2}}I$$

and

$$\int_T^\infty t^{1/2} (t^{-1}I - (\Gamma + t)^{-1}) dt \leq \frac{2\|\Gamma\|_\infty}{T^{1/2}} I.$$

Thus,

$$\left\| \int_T^\infty t^{1/2} (U(\Gamma_{\mathcal{N}} + t)^{-1} - (\Gamma + t)^{-1}U) \sigma_{\mathcal{N}}^{1/2} dt \right\|_2 \leq \frac{4\|\Gamma\|_\infty}{T^{1/2}},$$

since $U^*U \leq \text{Id}$ by Proposition 12.2.1, $\|\sigma_{\mathcal{N}}^{1/2}\|_2 = 1$, and $\|\Gamma_{\mathcal{N}}\|_\infty \leq \|\Gamma\|_\infty$ by Proposition 12.3.2. Therefore,

$$\left\| \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} \sigma_{\mathcal{N}}^{1/2} - \Gamma^{1/2} \sigma^{1/2} \right\|_2 \leq \frac{1}{\pi} T^{1/2} (\hat{S}_{\text{BS}}(\sigma \parallel \rho) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}} \parallel \rho_{\mathcal{N}}))^{1/2} + \frac{4\|\Gamma\|_\infty}{\pi T^{1/2}}.$$

Optimizing this expression with respect to T , we find the optimal bound

$$\left\| \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} \sigma_{\mathcal{N}}^{1/2} - \Gamma^{1/2} \sigma^{1/2} \right\|_2 \leq \frac{4\|\Gamma\|_\infty^{1/2}}{\pi} (\hat{S}_{\text{BS}}(\sigma \parallel \rho) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}} \parallel \rho_{\mathcal{N}}))^{1/4}.$$

Finally, rearranging the terms, we obtain Equation (12.13). ■

We have obtained a lower bound for the difference of BS-entropies in terms of one expression that already appeared in the previous section, in Corollary 12.2.8. Furthermore, we can find another lower bound for it with an expression that provides a measure of the recoverability of ρ in terms of the relation found in Theorem 12.2.2.

Lemma 12.3.4 — (BC19b).

Let \mathcal{M} be a matrix algebra with unital subalgebra \mathcal{N} . Let $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\rho > 0$, $\sigma > 0$ be two quantum states on \mathcal{M} . Then,

$$\left\| \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} \sigma_{\mathcal{N}}^{1/2} - \Gamma^{1/2} \sigma^{1/2} \right\|_2 \geq \frac{1}{2} \|\Gamma\|_\infty^{-1/2} \|\sigma^{-1}\|_\infty^{-1/2} \|\sigma \sigma_{\mathcal{N}}^{-1} \rho_{\mathcal{N}} - \rho\|_2.$$

Proof. Let us define

$$A := \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} \sigma_{\mathcal{N}}^{1/2} - \Gamma^{1/2} \sigma^{1/2}.$$

It holds that $\|\sigma_{\mathcal{N}}^{-1/2}\|_\infty \leq \|\sigma^{-1}\|_\infty^{1/2}$ by Jensen's operator inequality and the Russo-Dye theorem. Using the facts that $\|\sigma^{1/2}\|_2 = \|\sigma_{\mathcal{N}}^{1/2}\|_2 = 1$, on the one side we have

$$\begin{aligned} & \left\| \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}} - \Gamma \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \right\|_2 = \\ &= \left\| \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}} - \Gamma^{1/2} \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} + \Gamma^{1/2} \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} - \Gamma \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \right\|_2 \\ &\leq \left\| \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}} - \Gamma^{1/2} \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} \right\|_2 + \left\| \Gamma^{1/2} \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} - \Gamma \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \right\|_2 \\ &\leq \|A\|_2 \|\sigma_{\mathcal{N}}^{-1/2}\|_\infty (\|\Gamma_{\mathcal{N}}^{1/2}\|_\infty + \|\Gamma^{1/2}\|_\infty) \\ &\leq 2\|A\|_2 \|\sigma^{-1}\|_\infty^{1/2} \|\Gamma\|_\infty^{1/2}, \end{aligned}$$

where we have used Hölder's inequality and Proposition 12.3.2. On the other side, we get

$$\begin{aligned} \left\| \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}} - \Gamma \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \right\|_2 &= \left\| \sigma^{1/2} \sigma_{\mathcal{N}}^{-1} \rho_{\mathcal{N}} \sigma_{\mathcal{N}}^{-1/2} - \sigma^{-1/2} \rho \sigma_{\mathcal{N}}^{-1/2} \right\|_2 \\ &\geq \left\| \sigma \sigma_{\mathcal{N}}^{-1} \rho_{\mathcal{N}} - \rho \right\|_2. \end{aligned}$$

Therefore,

$$\|\sigma\sigma_{\mathcal{N}}^{-1}\rho_{\mathcal{N}} - \rho\|_2 \leq 2\|\Gamma\|_{\infty}^{1/2}\|\sigma^{-1}\|_{\infty}^{1/2}\left\|\sigma^{1/2}\sigma_{\mathcal{N}}^{-1/2}\Gamma_{\mathcal{N}}^{1/2}\sigma_{\mathcal{N}}^{1/2} - \Gamma^{1/2}\sigma^{1/2}\right\|_2.$$

■

Note that $\|\sigma^{-1}\|_{\infty}$ is nothing but the inverse of the minimum eigenvalue of σ . Finally, as a consequence of Theorem 12.3.3 and Lemma 12.3.4, we get the following corollary.

Corollary 12.3.5 — (BC19b).

Let \mathcal{M} be a matrix algebra with unital subalgebra \mathcal{N} . Let $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma > 0, \rho > 0$ be two quantum states on \mathcal{M} . Then,

$$\hat{S}_{\text{BS}}(\sigma\|\rho) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \geq \left(\frac{\pi}{8}\right)^4 \|\Gamma\|_{\infty}^{-4} \|\sigma^{-1}\|_{\infty}^{-2} \|\rho - \sigma\sigma_{\mathcal{N}}^{-1}\rho_{\mathcal{N}}\|_2^4. \quad (12.14)$$

R Remark 12.3.6

This result, in particular, constitutes another proof for the implication

$$\hat{S}_{\text{BS}}(\sigma\|\rho) = \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \implies \rho = \sigma\sigma_{\mathcal{N}}^{-1}\rho_{\mathcal{N}},$$

from Theorem 12.2.2. Indeed, we can deduce from the proof of this corollary the implications (1) \implies (3) \implies (2) in Corollary 12.2.8.

12.4 ON THE DATA PROCESSING INEQUALITY FOR MAXIMAL f -DIVERGENCES

In this section, we consider a more general setting than in the previous ones and, following the lines of [CV18], we provide a strengthened data processing inequality for maximal f -divergences. We consider operator convex functions $f : (0, \infty) \rightarrow \mathbb{R}$ whose transpose \tilde{f} is operator monotone decreasing. The transpose is operator convex by Proposition 4.4.5 and it is also monotone decreasing if f maps $(0, \infty)$ to itself by Theorem 4.4.4. Since the functions we consider here belong to a more general framework, we have to further assume the latter, although the aforementioned theorem shows that it is a reasonable assumption.

Moreover, we demand that the measure $\mu_{-\tilde{f}}$ of the transpose with negative sign is absolutely continuous with respect to Lebesgue measure and assume that there are $C > 0, \alpha \geq 0$ such that for every $T \geq 1$, the Radon-Nikodým derivative satisfies

$$\frac{d\mu_{-\tilde{f}}(t)}{dt} \geq (CT^{2\alpha})^{-1}$$

almost everywhere (with respect to Lebesgue measure) for all $t \in [1/T, T]$. Moreover, we impose the condition that our states $\sigma, \rho > 0$ are such that

$$\left(\frac{(2\alpha + 1)\sqrt{C}}{4} \frac{(\hat{S}_f(\sigma\|\rho) - \hat{S}_f(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}))^{1/2}}{1 + \|\Gamma\|_{\infty}} \right)^{\frac{1}{1+\alpha}} \leq 1. \quad (12.15)$$

The main result of this section is the following:

Theorem 12.4.1 — STABILITY OF THE DPI FOR MAXIMAL f -DIVERGENCES, (BC19b).

Let \mathcal{M} be a matrix algebra with unital subalgebra \mathcal{N} . Let $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma > 0, \rho > 0$ be two quantum states on \mathcal{M} and let $f : (0, \infty) \rightarrow \mathbb{R}$ be an operator convex function with transpose \tilde{f} . We assume that \tilde{f} is operator monotone decreasing and such that the measure $\mu_{-\tilde{f}}$ that appears in Theorem 4.4.2 is absolutely continuous with respect to Lebesgue measure. Moreover, we assume that for every $T \geq 1$, there exist constants $\alpha \geq 0, C > 0$ satisfying $d\mu_{-\tilde{f}}(t)/dt \geq (CT^{2\alpha})^{-1}$ for all $t \in [1/T, T]$ and such that Equation (12.15) holds. Then, there is a constant $K_\alpha > 0$ such that

$$\hat{S}_f(\sigma \| \rho) - \hat{S}_f(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}}) \geq \frac{K_\alpha}{C} (1 + \|\Gamma\|_\infty)^{-(4\alpha+2)} \left\| \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} \sigma_{\mathcal{N}}^{1/2} - \Gamma^{1/2} \sigma^{1/2} \right\|_2^{4(\alpha+1)}. \quad (12.16)$$

Proof. Recall that, given an operator convex function f with transpose \tilde{f} ,

$$\hat{S}_f(\sigma \| \rho) = \hat{S}_{\tilde{f}}(\rho \| \sigma) = \text{tr} \left[\sigma^{1/2} \tilde{f}(\Gamma) \sigma^{1/2} \right]$$

by Proposition 12.1.9. By assumption, \tilde{f} is operator monotone decreasing. Thus, by virtue of Theorem 4.4.2, $-\tilde{f}$ can be written as

$$-\tilde{f}(\lambda) = \alpha + \beta \lambda + \int_0^\infty \left(\frac{t}{t^2 + 1} - \frac{1}{t + \lambda} \right) d\mu_{-\tilde{f}}(t),$$

with $\alpha \in \mathbb{R}, \beta \geq 0$ and $\mu_{-\tilde{f}}$ a positive measure on $(0, \infty)$ such that

$$\int_0^\infty \frac{1}{t^2 + 1} d\mu_{-\tilde{f}}(t) < \infty.$$

Hence, it is clear that

$$\begin{aligned} \hat{S}_f(\sigma \| \rho) &= \left\langle \sigma^{1/2}, \tilde{f}(\Gamma) \sigma^{1/2} \right\rangle \\ &= \left\langle \sigma^{1/2}, \left(-\alpha - \beta \Gamma + \int_0^\infty \left((\Gamma + t)^{-1} - \frac{t}{t^2 + 1} I \right) d\mu_{-\tilde{f}}(t) \right) \sigma^{1/2} \right\rangle \\ &= -\alpha - \beta + \int_0^\infty \left\langle \sigma^{1/2}, \left((\Gamma + t)^{-1} - \frac{t}{t^2 + 1} I \right) \sigma^{1/2} \right\rangle d\mu_{-\tilde{f}}(t) \\ &\geq -\alpha - \beta + \int_0^\infty \left\langle \sigma_{\mathcal{N}}^{1/2}, \left((\Gamma_{\mathcal{N}} + t)^{-1} - \frac{t}{t^2 + 1} I \right) \sigma_{\mathcal{N}}^{1/2} \right\rangle d\mu_{-\tilde{f}}(t) \\ &= \hat{S}_f(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}}), \end{aligned}$$

where the inequality in the fourth line follows from Proposition 12.2.1 and Jensen's operator inequality (point (3) in Theorem 4.4.6). Note that the difference of maximal f -divergences is given by

$$\hat{S}_f(\sigma \| \rho) - \hat{S}_f(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}}) = \int_0^\infty \left(\left\langle \sigma^{1/2}, (\Gamma + t)^{-1} \sigma^{1/2} \right\rangle - \left\langle \sigma_{\mathcal{N}}^{1/2}, (\Gamma_{\mathcal{N}} + t)^{-1} \sigma_{\mathcal{N}}^{1/2} \right\rangle \right) d\mu_{-\tilde{f}}(t),$$

and, recalling that $U \sigma_{\mathcal{N}}^{1/2} = \sigma^{1/2}$, the difference between the infinitesimal terms in the integrals was studied in Lemma 12.3.1, obtaining

$$\left\langle \sigma_{\mathcal{N}}^{1/2}, (U^*(\Gamma + t)^{-1} U - (\Gamma_{\mathcal{N}} + t)^{-1}) \sigma_{\mathcal{N}}^{1/2} \right\rangle \geq t \left\| (U(\Gamma_{\mathcal{N}} + t)^{-1} - (\Gamma + t)^{-1} U) \sigma_{\mathcal{N}}^{1/2} \right\|_2^2,$$

Following the proof of Theorem 12.3.3, we infer that

$$\left\| \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} \sigma_{\mathcal{N}}^{1/2} - \Gamma^{1/2} \sigma^{1/2} \right\|_2 = \frac{1}{\pi} \left\| \int_0^\infty t^{1/2} (U(\Gamma_{\mathcal{N}} + t)^{-1} - (\Gamma + t)^{-1} U) \sigma_{\mathcal{N}}^{1/2} dt \right\|_2,$$

and we can now split the right hand side into three parts (contrary to the proof of Theorem 12.3.3, where we only split it in two):

$$\begin{aligned} \left\| \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} \sigma_{\mathcal{N}}^{1/2} - \Gamma^{1/2} \sigma^{1/2} \right\|_2 &\leq \underbrace{\frac{1}{\pi} \int_0^{1/T} t^{1/2} \left\| (U(\Gamma_{\mathcal{N}} + t)^{-1} - (\Gamma + t)^{-1} U) \sigma_{\mathcal{N}}^{1/2} \right\|_2 dt}_{(*)1} \\ &\quad + \underbrace{\frac{1}{\pi} \int_{1/T}^T t^{1/2} \left\| (U(\Gamma_{\mathcal{N}} + t)^{-1} - (\Gamma + t)^{-1} U) \sigma_{\mathcal{N}}^{1/2} \right\|_2 dt}_{(*)2} \\ &\quad + \underbrace{\frac{1}{\pi} \left\| \int_T^\infty t^{1/2} (U(\Gamma_{\mathcal{N}} + t)^{-1} - (\Gamma + t)^{-1} U) \sigma_{\mathcal{N}}^{1/2} dt \right\|_2}_{(*)3}. \end{aligned}$$

Here, we assume that $T \geq 1$. Let us study each one of these terms separately: For the first one, we have

$$(*)1 \leq \frac{2}{\pi} \int_0^{1/T} t^{-1/2} dt = \frac{4}{\pi T^{1/2}},$$

since

$$\left\| (U(\Gamma_{\mathcal{N}} + t)^{-1} - (\Gamma + t)^{-1} U) \sigma_{\mathcal{N}}^{1/2} \right\|_2 \leq 2t^{-1}.$$

The last term is bounded using the same reasoning as in the proof of Theorem 12.3.3. Thus, we have

$$(*)3 \leq \frac{4 \|\Gamma\|_\infty}{\pi T^{1/2}}.$$

The second term, however, introduces something that had not appeared on the main result of the previous section. Indeed, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} (*)2 &\leq \frac{1}{\pi} \left(T - \frac{1}{T} \right)^{1/2} \left(\int_{1/T}^T t \left\| (U(\Gamma_{\mathcal{N}} + t)^{-1} - (\Gamma + t)^{-1} U) \sigma_{\mathcal{N}}^{1/2} \right\|_2^2 dt \right)^{1/2} \\ &\leq \frac{T^{1/2}}{\pi} \sqrt{C} T^\alpha \left(\int_{1/T}^T t \left\| (U(\Gamma_{\mathcal{N}} + t)^{-1} - (\Gamma + t)^{-1} U) \sigma_{\mathcal{N}}^{1/2} \right\|_2^2 d\mu_{-\hat{f}}(t) \right)^{1/2} \\ &\leq \frac{T^{1/2}}{\pi} \sqrt{C} T^\alpha (\hat{S}_f(\sigma \| \rho) - \hat{S}_f(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}}))^{1/2}. \end{aligned}$$

Let us assume that $\hat{S}_f(\sigma \| \rho) - \hat{S}_f(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}}) > 0$. Considering the three bounds together and optimizing over T , we find that the minimum is achieved for

$$T = \left(\frac{4}{(2\alpha + 1) \sqrt{C}} \frac{1 + \|\Gamma\|_\infty}{(\hat{S}_f(\sigma \| \rho) - \hat{S}_f(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}}))^{1/2}} \right)^{\frac{1}{1+\alpha}}.$$

We note that indeed $T \geq 1$ by Equation (12.15). Inserting the optimal T , we obtain

$$\begin{aligned} &\left\| \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} \sigma_{\mathcal{N}}^{1/2} - \Gamma^{1/2} \sigma^{1/2} \right\|_2 \\ &\leq (K_\alpha)^{-\frac{1}{4(\alpha+1)}} (1 + \|\Gamma\|_\infty)^{\frac{2\alpha+1}{2\alpha+2}} C^{\frac{1}{4(\alpha+1)}} (\hat{S}_f(\sigma \| \rho) - \hat{S}_f(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}}))^{\frac{1}{4(\alpha+1)}}, \end{aligned}$$

and rearranging the terms, we get Equation (12.16). Here,

$$K_\alpha = \left(\frac{2\alpha+1}{2\alpha+2} \right)^{4(\alpha+1)} (2\alpha+1)^{-2} 4^{-(4\alpha+2)} \pi^{4(\alpha+1)}.$$

This bound is also valid for $\hat{S}_f(\sigma\|\rho) - \hat{S}_f(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) = 0$, since in this case we can make the upper bound arbitrarily small by choosing T arbitrarily large. ■

Lemma 12.3.4 can also be used to get another bound for the difference of maximal f -divergences in terms of the BS recovery map applied to ρ .

Corollary 12.4.2 — (BC19b).

Let \mathcal{M} be a matrix algebra with unital subalgebra \mathcal{N} . Let $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma > 0$, $\rho > 0$ be two quantum states on \mathcal{M} and let $f : (0, \infty) \rightarrow \mathbb{R}$ be an operator convex function with transpose \tilde{f} . We assume that \tilde{f} is operator monotone decreasing and such that the measure $\mu_{-\tilde{f}}$ that appears in Theorem 4.4.2 is absolutely continuous with respect to Lebesgue measure. Moreover, we assume that for every $T \geq 1$, there exist constants $\alpha \geq 0$, $C > 0$ satisfying $d\mu_{-\tilde{f}}(t)/dt \geq (CT^{2\alpha})^{-1}$ for all $t \in [1/T, T]$ and such that Equation (12.15) holds. Then, there is a constant $L_\alpha > 0$ such that

$$\begin{aligned} \hat{S}_f(\sigma\|\rho) - \hat{S}_f(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \\ \geq \frac{L_\alpha}{C} (1 + \|\Gamma\|_\infty)^{-(4\alpha+2)} \|\Gamma\|_\infty^{-(2\alpha+2)} \|\sigma^{-1}\|_\infty^{-(2\alpha+2)} \|\rho - \sigma\sigma_{\mathcal{N}}^{-1}\rho_{\mathcal{N}}\|_2^{4(\alpha+1)}. \end{aligned} \quad (12.17)$$

As a consequence of Theorem 12.4.1, we get the following strengthening of the data processing inequality for maximal f -divergences for particular operator convex functions. The first one concerns the BS-entropy. In this case, $f(x) = x \log x$, $\tilde{f}(x) = -\log x$, $\alpha = 0$ and $C = 1$.

Corollary 12.4.3 — (BC19b).

Let \mathcal{M} be a matrix algebra with unital subalgebra \mathcal{N} . Let $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma > 0$, $\rho > 0$ be two quantum states on \mathcal{M} such that $\hat{S}_{\text{BS}}(\sigma\|\rho) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \leq 4(\|\Gamma\|_\infty + 1)$. Then,

$$\hat{S}_{\text{BS}}(\sigma\|\rho) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \geq \left(\frac{\pi}{8} \right)^4 (1 + \|\Gamma\|_\infty)^{-2} \|\Gamma\|_\infty^{-2} \|\sigma^{-1}\|_\infty^{-2} \|\rho - \sigma\sigma_{\mathcal{N}}^{-1}\rho_{\mathcal{N}}\|_2^4. \quad (12.18)$$

Note that Equation (12.18) is a bit less tight than Equation (12.14), although the results are comparable. The next corollary deals with the data processing inequality of maximal f -divergences associated to power functions.

Corollary 12.4.4 — (BC19b).

Let \mathcal{M} be a matrix algebra with unital subalgebra \mathcal{N} . Let $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma > 0$, $\rho > 0$ be two quantum states on \mathcal{M} and take $f_\beta(x) = -x^{1-\beta}$, for $0 < \beta < 1$. Then, $C = \frac{\pi}{\sin \pi \beta}$, $\alpha = \beta/2$ and if Equation (12.15) holds, we have:

$$\begin{aligned} \hat{S}_{f_\beta}(\sigma \| \rho) - \hat{S}_{f_\beta}(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}}) \\ \geq L_{\beta/2} \frac{\sin \pi \beta}{\pi} (1 + \|\Gamma\|_\infty)^{-2(\beta+1)} \|\Gamma\|_\infty^{-(\beta+2)} \|\sigma^{-1}\|_\infty^{-(\beta+2)} \|\rho - \sigma \sigma_{\mathcal{N}}^{-1} \rho_{\mathcal{N}}\|_2^{2\beta+4}. \end{aligned} \quad (12.19)$$

Proof. This follows straight from the facts $\tilde{f}(x) = -x^\beta$ and that [CV18, Example 3]

$$d\mu_{f_\beta}(t) = \frac{\sin \pi \beta}{\pi} t^\beta dt.$$

An application of Theorem 12.4.1 and Lemma 12.3.4 yields

$$L_{\beta/2} = \frac{1}{4} \left(\frac{\beta+1}{\beta+2} \right)^{2\beta+4} (\beta+1)^{-2} 8^{-2(\beta+1)} \pi^{2\beta+4}.$$

■

12.5 EXTENSION OF THE PREVIOUS RESULTS TO GENERAL QUANTUM CHANNELS

The purpose of this section is to present an extension of the main results obtained in Sections 12.3 and 12.4 to general quantum channels. To this end, we will adopt the following strategy: First, we will see that our results extend to states which are not full rank. Then, we will use a Stinespring dilation to lift our results to arbitrary quantum channels. In this case, the theorem corresponding to the main result of Section 12.3 reads as follows:

Theorem 12.5.1 — STRENGTHENED DPI FOR THE BS-ENTROPY FOR GENERAL CHANNELS, (BC19b).

Let \mathcal{M} and \mathcal{N} be two matrix algebras and let $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ be a completely positive trace-preserving map with V the isometry from a Stinespring dilation of \mathcal{T} (Theorem 4.4.9). Let σ , ρ be two quantum states on \mathcal{M} such that $\rho^0 = \sigma^0$. Then

$$\hat{S}_{\text{BS}}(\sigma \| \rho) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{T}} \| \rho_{\mathcal{T}}) \geq \left(\frac{\pi}{4} \right)^4 \|\Gamma\|_\infty^{-2} \left\| V \sigma^{1/2} V^* \sigma_{\mathcal{T}}^{-1/2} \Gamma_{\mathcal{T}}^{1/2} \sigma_{\mathcal{T}}^{1/2} \otimes I - V \Gamma^{1/2} \sigma^{1/2} V^* \right\|_2^4. \quad (12.20)$$

Here, σ^{-1} and $\sigma_{\mathcal{T}}^{-1}$ are the Moore-Penrose inverses if the states are not invertible. Moreover, we have

$$\hat{S}_{\text{BS}}(\sigma \| \rho) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{T}} \| \rho_{\mathcal{T}}) \geq \left(\frac{\pi}{8} \right)^4 \|\Gamma\|_\infty^{-4} \|\sigma_{\mathcal{T}}^{-1}\|_\infty^{-2} \|\sigma_{\mathcal{T}}^* (\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}}) - \rho\|_2^4. \quad (12.21)$$

Proof. Let us first justify that the quantities that appear in Equation (12.20) are well-defined for non full-rank states for the case in which the map considered is a trace-preserving conditional expectation. Let us recall that the BS-entropy for non full-rank states σ , ρ is given by

$$\hat{S}_{\text{BS}}(\sigma \| \rho) = \lim_{\varepsilon \searrow 0} \hat{S}_{\text{BS}}(\sigma + \varepsilon I \| \rho + \varepsilon I).$$

By virtue of Douglas' lemma [Dou66, Theorem 1] $\rho^0 = \sigma^0$ implies $\mathcal{T}(\rho)^0 = \mathcal{T}(\sigma)^0$ for every positive map \mathcal{T} . Hence, it follows from [HM17, Proposition 3.29] that the left-hand side

of Equation (12.20) is also finite for non full-rank states. Furthermore, given $a, b > 0$ and $\sigma > 0$, $\rho > 0$, we can easily see that

$$\hat{S}_{\text{BS}}(a\sigma \| b\rho) = a\hat{S}_{\text{BS}}(\sigma \| \rho) + a\log\left(\frac{a}{b}\right).$$

Given a conditional expectation $\mathcal{E} : \mathcal{M} \rightarrow \mathcal{N}$, we define

$$\sigma^\varepsilon := \frac{\sigma + \varepsilon I}{1 + \varepsilon d}, \quad \rho^\varepsilon := \frac{\rho + \varepsilon I}{1 + \varepsilon d}, \quad \sigma_{\mathcal{N}}^\nu := \frac{\sigma_{\mathcal{N}} + \nu I}{1 + \nu d}, \quad \rho_{\mathcal{N}}^\nu := \frac{\rho_{\mathcal{N}} + \nu I}{1 + \nu d}.$$

Here, we have assumed that the identity in \mathcal{M} has trace $d \in \mathbb{N}$. By the above, we have

$$\begin{aligned} \hat{S}_{\text{BS}}(\sigma \| \rho) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}}) &= \lim_{\varepsilon \searrow 0} \hat{S}_{\text{BS}}(\sigma + \varepsilon I \| \rho + \varepsilon I) - \lim_{\nu \searrow 0} \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}} + \nu I \| \rho_{\mathcal{N}} + \nu I) \\ &= \lim_{\varepsilon \searrow 0} \lim_{\nu \searrow 0} [(1 + d\varepsilon)\hat{S}_{\text{BS}}(\sigma^\varepsilon \| \rho^\varepsilon) - (1 + d\nu)\hat{S}_{\text{BS}}(\sigma_{\mathcal{N}}^\nu \| \rho_{\mathcal{N}}^\nu)], \end{aligned}$$

where we can choose $\varepsilon = \nu$ in particular. For $\sigma^\varepsilon, \rho^\varepsilon$, Equation (12.13) reads as

$$\begin{aligned} \hat{S}_{\text{BS}}(\sigma^\varepsilon \| \rho^\varepsilon) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}}^\varepsilon \| \rho_{\mathcal{N}}^\varepsilon) \\ \geq \left(\frac{\pi}{4}\right)^4 \|\Gamma^\varepsilon\|_\infty^{-2} \left\| (\sigma^\varepsilon)^{1/2} (\sigma_{\mathcal{N}}^\varepsilon)^{-1/2} (\Gamma_{\mathcal{N}}^\varepsilon)^{1/2} (\sigma_{\mathcal{N}}^\varepsilon)^{1/2} - (\Gamma^\varepsilon)^{1/2} (\sigma^\varepsilon)^{1/2} \right\|_2^4, \end{aligned}$$

where $\Gamma^\varepsilon := (\sigma^\varepsilon)^{-1/2} \rho^\varepsilon (\sigma^\varepsilon)^{-1/2}$ and $\Gamma_{\mathcal{N}}^\varepsilon := (\sigma_{\mathcal{N}}^\varepsilon)^{-1/2} \rho_{\mathcal{N}}^\varepsilon (\sigma_{\mathcal{N}}^\varepsilon)^{-1/2}$. The only thing left to do is to write the right-hand side of the expression above in terms of σ and ρ . However, expanding σ^ε and ρ^ε in the right basis, if we write $P = \sigma^0$, one can show that Γ^ε converges to $\Gamma_P \oplus I$, where $\Gamma_P = (\sigma|_P)^{-1/2} \rho|_P (\sigma|_P)^{-1/2}$ and we identify P with the subspace it projects onto. Moreover, we can see using the spectral decomposition of σ that $\|\Gamma_P\|_\infty \geq 1$, such that

$$\|\Gamma_P \oplus I\|_\infty = \|\Gamma_P\|_\infty$$

and

$$\lim_{\varepsilon \searrow 0} (\Gamma^\varepsilon)^{1/2} = \Gamma_P^{1/2} \oplus I.$$

Similar considerations lead to

$$\hat{S}_{\text{BS}}(\sigma \| \rho) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}}) \geq \left(\frac{\pi}{4}\right)^4 \|\Gamma\|_\infty^{-2} \left\| \sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} \sigma_{\mathcal{N}}^{1/2} - \Gamma^{1/2} \sigma^{1/2} \right\|_2^4, \quad (12.22)$$

where the states σ and ρ are not necessarily full-rank anymore, and thus the inverses are now Moore-Penrose inverses. Now, following the steps of [Wil18], we are in position to apply Stinespring's dilation theorem (Theorem 4.4.9).

Given ω and τ states on \mathcal{M} such that $\omega^0 = \tau^0$, let us define

$$\sigma := V \omega V^*,$$

$$\rho := V \tau V^*.$$

Then, it is clear that $\mathcal{E}(\sigma) = \mathcal{T}(\omega) \otimes I/s$ and $\mathcal{E}(\rho) = \mathcal{T}(\tau) \otimes I/s$ for $\mathcal{E} = \text{tr}_{\mathcal{V}}[\cdot] \otimes I/s$ and $\dim \mathcal{V} = s$. Since \mathcal{E} is a conditional expectation, the Inequality (12.22) holds for it and σ and ρ defined as above, yielding:

$$\hat{S}_{\text{BS}}(\omega \| \tau) - \hat{S}_{\text{BS}}(\mathcal{T}(\omega) \| \mathcal{T}(\tau)) \geq \left(\frac{\pi}{4}\right)^4 \|\Gamma\|_\infty^{-2} \left\| V \omega^{1/2} V^* \omega_{\mathcal{T}}^{-1/2} \Gamma_{\mathcal{T}}^{1/2} \omega_{\mathcal{T}}^{1/2} \otimes I - V \Gamma^{1/2} \omega^{1/2} V^* \right\|_2^4,$$

where here we define $\Gamma := \omega^{-1/2} \tau \omega^{-1/2}$ and $\Gamma_{\mathcal{T}} := \omega_{\mathcal{T}}^{-1/2} \tau_{\mathcal{T}} \omega_{\mathcal{T}}^{-1/2}$ for $\omega_{\mathcal{T}} := \mathcal{T}(\omega)$ and $\tau_{\mathcal{T}} := \mathcal{T}(\tau)$, since

$$\begin{aligned}\hat{S}_{\text{BS}}(\sigma || \rho) &= \hat{S}_{\text{BS}}(\omega || \tau), \\ \hat{S}_{\text{BS}}(\mathcal{E}(\sigma) || \mathcal{E}(\rho)) &= \hat{S}_{\text{BS}}(\mathcal{T}(\omega) || \mathcal{T}(\tau)),\end{aligned}$$

for the terms in the left-hand side. Moreover,

$$\begin{aligned}\left\| \sigma^{-1/2} \rho \sigma^{-1/2} \right\|_{\infty} &= \left\| V \omega^{-1/2} V^* V \tau V^* V \omega^{-1/2} V^* \right\|_{\infty} \\ &= \left\| \Gamma \right\|_{\infty}\end{aligned}$$

and

$$\begin{aligned}& \left\| \sigma^{1/2} \mathcal{E}(\sigma)^{-1/2} \left(\mathcal{E}(\sigma)^{-1/2} \mathcal{E}(\rho) \mathcal{E}(\sigma)^{-1/2} \right)^{1/2} \mathcal{E}(\sigma)^{1/2} - \left(\sigma^{-1/2} \rho \sigma^{-1/2} \right)^{1/2} \sigma^{1/2} \right\|_2 \\ &= \left\| V \omega^{1/2} V^* \omega_{\mathcal{T}}^{-1/2} \left(\omega_{\mathcal{T}}^{-1/2} \tau_{\mathcal{T}} \omega_{\mathcal{T}}^{-1/2} \right)^{1/2} \omega_{\mathcal{T}}^{1/2} \otimes I - V \left(\omega^{-1/2} \tau \omega^{-1/2} \right)^{1/2} \omega^{1/2} V^* \right\|_2.\end{aligned}$$

for the terms in the right hand-side, where we have only used the fact that V is an isometry. The second assertion follows from minor adjustments to the proof of Lemma 12.3.4. ■

Before we can continue, we need to prove that we obtain $\hat{S}_f(\sigma || \rho)$ for non-invertible σ, ρ from a limit of states.

Proposition 12.5.2 — (BC19b).

Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a matrix algebra for a Hilbert space \mathcal{H} of dimension d and let σ, ρ be states on \mathcal{M} such that $\rho^0 = \sigma^0$. Then,

$$\hat{S}_f(\sigma || \rho) = \lim_{\varepsilon \searrow 0} \hat{S}_f((\sigma + \varepsilon I)/(1 + d\varepsilon) || (\rho + \varepsilon I)/(1 + d\varepsilon))$$

for every operator convex function $f : (0, \infty) \rightarrow \mathbb{R}$.

Proof. Proposition 3.29 of [HM17] asserts that $\hat{S}_f(\sigma || \rho)$ is finite. Let

$$P_f(A, B) := B^{1/2} f(B^{-1/2} A B^{-1/2}) B^{1/2}$$

for positive definite $A, B \in \mathcal{B}(\mathcal{H})$. This is the non-commutative perspective function defined in [HM17, Equation 2.7]. Corollary 3.28 of [HM17] shows that for states such that $\rho^0 = \sigma^0$

$$\hat{S}_f(\sigma || \rho) = \lim_{n \rightarrow \infty} \hat{S}_f(\sigma + K_n || \rho + K_n),$$

where $K_n \in \mathcal{B}(\mathcal{H})$ is any sequence with $K_n \rightarrow 0$ such that $K_n \geq 0$ and $\sigma + K_n, \rho + K_n > 0$ for every $n \in \mathbb{N}$. Thus, in particular we can choose $K_n = \varepsilon_n I / (1 + \varepsilon_n d)$ and $\varepsilon_n \rightarrow 0$. ■

Using the same ideas that appear in the proof of the previous result together with Proposition 12.5.2, we can also obtain from Theorem 12.4.1 the following analogous result for general quantum channels.

Theorem 12.5.3 — STRENGTHENED DPI FOR MAXIMAL f -DIVERGENCES FOR CHANNELS, (BC19b).

Let \mathcal{M} and \mathcal{N} be two matrix algebras and let $\mathcal{T} : \mathcal{M} \rightarrow \mathcal{N}$ be a completely positive trace-preserving map with V the isometry from its Stinespring dilation (Theorem 4.4.9). Let σ, ρ be two quantum states on \mathcal{M} such that $\rho^0 = \sigma^0$ and let $f : (0, \infty) \rightarrow \mathbb{R}$ be an operator convex function with transpose \tilde{f} . We assume that \tilde{f} is operator monotone decreasing and such that the measure $\mu_{-\tilde{f}}$ that appears in Theorem 4.4.2 is absolutely continuous with respect to Lebesgue measure. Moreover, we assume that for every $T \geq 1$, there exist constants $\alpha \geq 0$, $C > 0$ satisfying $d\mu_{-\tilde{f}}(t)/dt \geq (CT^{2\alpha})^{-1}$ for all $t \in [1/T, T]$ and such that Equation (12.15) holds. Then, there is a constant $K_\alpha > 0$ such that

$$\begin{aligned} \hat{S}_f(\sigma \| \rho) - \hat{S}_f(\sigma_{\mathcal{T}} \| \rho_{\mathcal{T}}) \\ \geq \frac{K_\alpha}{C} (1 + \|\Gamma\|_\infty)^{-(4\alpha+2)} \left\| V \sigma^{1/2} V^* \sigma_{\mathcal{T}}^{-1/2} \Gamma_{\mathcal{T}}^{1/2} \sigma_{\mathcal{T}}^{1/2} \otimes I - V \Gamma^{1/2} \sigma^{1/2} V^* \right\|_2^{4(\alpha+1)}. \end{aligned}$$

Furthermore, there is another constant $L_\alpha > 0$ such that

$$\begin{aligned} \hat{S}_f(\sigma \| \rho) - \hat{S}_f(\sigma_{\mathcal{T}} \| \rho_{\mathcal{T}}) \\ \geq \frac{L_\alpha}{C} (1 + \|\Gamma\|_\infty)^{-(4\alpha+2)} \|\Gamma\|_\infty^{-(2\alpha+2)} \|\sigma_{\mathcal{T}}^{-1}\|_\infty^{-(2\alpha+2)} \|\rho - \sigma_{\mathcal{T}}^* (\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}})\|_2^{4(\alpha+1)}. \end{aligned}$$

Here, we consider again Moore-Penrose inverses if the states are not invertible.

R Remark 12.5.4

Note that the procedure followed to extend Theorems 12.3.3 and 12.4.1 to Theorems 12.5.1 and 12.5.3, respectively, consists of two main ingredients: The extension of our previous results to not necessarily full-rank states followed by Stinespring's dilation theorem. Analogously to what we have done in this section, this procedure can be also applied to the setting presented in [CV17] and [CV18] to extend the main results therein to general quantum channels. In particular, Equation (11.34) holds for general quantum channels.



CONCLUSIONS AND OPEN PROBLEMS

In this thesis, we have studied quantum dissipative evolutions and the speed of convergence to their equilibrium state. More specifically, we have addressed the problem of finding static sufficient conditions on the fixed point of a quantum dissipative evolution so that the convergence is fast enough, that is the system satisfies rapid mixing. Since a positive log-Sobolev constant provides a bound for the mixing time of an evolution that might scale logarithmically with the system size under some conditions on the fixed point, the problem of proving rapid mixing can be reduced to prove the positivity of this constant.

Therefore, in this thesis we have focused on finding static sufficient conditions on the fixed point of a quantum dissipative evolution so that its generator has a positive log-Sobolev constant. The classical analogous problem has been studied in the past by following several strategies, one of the most fruitful of which uses strongly results of quasi-factorization of the entropy. Hence, first we have aimed to provide a quantum analogue for this strategy, based on results of quasi-factorization of the relative entropy.

Subsequently, we have tested the conceived strategy for the heat-bath and Davies dynamics under several different conditions of clustering of correlations on the fixed points of the evolutions. After proving some results of weak and strong quasi-factorization from certain notions of conditional relative entropy that we have previously introduced, we have used them to obtain results of positivity for the log-Sobolev constant associated to the heat-bath and the Davies generators, respectively, under some conditions of clustering of correlations on the fixed points of the evolutions.

The outcomes of this thesis constitute altogether a large step towards the solution to the problem of proving positivity for log-Sobolev constants under certain conditions on the fixed points of the evolutions, since not only we provide positive results for this problem for certain dynamics, but we also provide a general strategy that extends its classical analogue and works at least in the cases studied, providing new interesting results in that line.

However, there are also many natural problems that arise from this thesis, both in the line of the results of quasi-factorization as well as in the line of log-Sobolev inequalities. We will state some of them below and show possible approaches to tackle them.

Moreover, the field of log-Sobolev inequalities has (or might have) profound implications in many directions, not just to serve as a sufficient condition for a quantum system to satisfy rapid

This is a picture of the beautiful gardens of the campus of *Caltech*, in Pasadena (California), where I was visiting in April 2019.

mixing. Some lines of work that might be pursued by the candidate after the completion of this thesis, either to extend results included here or to apply them (or the techniques and ideas used to obtain them) to different settings will be discussed below.

FUTURE WORK

The future lines of research that arise from this thesis are numerous. For simplicity, we can classify them mainly into two classes: Extensions of results appearing here or application of them to different problems.

Concerning possible extensions of results from this thesis, there are several directions in which we can proceed now and which we collect below.

- *Finding examples of conditions of clustering of correlations.* One of the weakest points of the results of positivity of log-Sobolev constant appearing in this thesis is the lack of examples for which the assumed conditions of clustering of correlations hold. Therefore, we aim to continue studying physical systems for which any of these conditions might hold and prove it. A better understanding of the clustering conditions assumed for classical spin systems might help in this direction.
- *Weakening sufficient conditions.* Closely related to the previous point, an interesting problem we consider is to weaken the conditions that we assume for the strategy to prove positivity of the log-Sobolev constant. If successful, this might leave more space for quantum systems to satisfy the new (possibly-not-so-strong) conditions.
- *Enlarging the class of systems for which our results work.* Another line of work we are pursuing now is to try to extend the results mentioned in this text to larger classes. In particular, one result of this kind would be the extension of the positivity for the log-Sobolev constant associated to the heat-bath generator to systems with dimension larger than 1.
- *Understanding connections between different conditions of clustering of correlations.* Finally, a better understanding of the connections between different conditions of clustering of correlations might lead to the existence of more examples of quantum systems for which our results hold. In particular, we aim to find a relation between the mixing condition assumed for the heat-bath dynamics and the conditional \mathbb{L}_1 -clustering of correlations considered for the Davies dynamics, since in the classical case both reduce to the same condition.

Let us turn now to applications. The results derived in this thesis, or the techniques used to do so, can be applied to interesting problems from different fields, as we mention below.

- *Quantum circuits.* Logarithmic Sobolev constants associated to dynamics generated by a quantum channel that models some kind of noise (such that the depolarizing channel) in a quantum circuit can be used to estimate the amount of noise the circuit can present, and thus, the fidelity of the circuit. This approach was born quite recently [BC19a], but the results that might be obtained from it look really promising.
- *Mixing rates of divergences.* The procedure followed to estimate the mixing time of a dissipative evolution in terms of the mixing rate given by the entropy (the log-Sobolev constant) may be extended to a more general framework, such as the one generated by Rényi divergences or even f -divergences. For that, first we would have to prove a result of quasi-factorization of the respective divergence in terms of some conditional ones and subsequently we would need to lift the results presented in this thesis to the new setting.

- *Quantum capacities of channels.* The definition of the log-Sobolev constant appears after differentiating the relative entropy. If instead we consider initially another quantity, such as the conditional entropy, it is easy to prove that the optimal constant corresponding to the functional inequality which is analogous to the log-Sobolev inequality in the latter case provides an estimation of the quantum capacity of a channel [BCR19a], following the same steps than to prove the upper bound for the mixing time of a dissipative evolution in terms of the log-Sobolev constant. This approach might lead to better estimates on channel capacities than the previously-known ones.

OPEN PROBLEMS

Let us turn now to some open problems that arise from this thesis. One of the main question that yields from the work on conditional relative entropies is related to the main result of Section 7.3.

In Chapter 9, we show that a result of quasi-factorization of the relative entropy, when the second state is a tensor product, is the key tool to prove that the heat-bath dynamics, with product fixed point, has a positive log-Sobolev constant. The same strategy might be followed to obtain a positive log-Sobolev constant for the heat-bath dynamics, in a more general setting, from the stronger result of quasi-factorization presented in Theorem 7.3.1, under the assumption of a decay of correlations in the fixed point, as we show in Chapter 10. However, whether this result can be used with that objective in more than $1D$ is left as an open problem.

Problem 12.5.5 *Use the result of quasi-factorization of the relative entropy in terms of conditional relative entropies of Section 7.3 to obtain positive log-Sobolev constant for the heat-bath dynamics in dimension greater than 1, with a general fixed point σ .*

Considering the approach followed for the Davies dynamics, it is likely that, to get a better result for the heat-bath, we need to employ the conditional relative entropy by expectations instead of the usual one. For that, we first need to improve the result of quasi-factorization for this family of conditional relative entropies, Theorem 7.4.1.

Problem 12.5.6 *Improve the result of quasi-factorization of the conditional relative entropy by expectations of Theorem 7.4.1, by improving the bound that we obtained for the error term.*

Concerning the definition of conditional relative entropy presented here, we have shown several clues that allow us to think that the definition is reasonable. However, there is some space to possibly improve it, in some sense, so that we can obtain results of quasi-factorization more easily, for example.

Problem 12.5.7 *Improve the definition of conditional relative entropy. One idea to do that could be to add the property proven in equation (7.1) to the definition.*

Furthermore, is there any possible axiomatic definition for conditional relative entropy from which we can immediately obtain results of quasi-factorization?

Moreover, in Section 6.4, we have compared the definitions of conditional relative entropy and conditional relative entropy by expectations. On the one side, we have shown several cases where they coincide, and on the other side, we have seen that this cannot hold always. We leave the possibility of studying in general for which cases both expressions are the same as an open problem:

Problem 12.5.8 *Characterize for which states $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$, the following holds:*

$$D_A(\rho_{AB} || \sigma_{AB}) = D_A^E(\rho_{AB} || \sigma_{AB}),$$

or, at least, find more examples where this equality holds.

Now, when introducing the result of positivity of the log-Sobolev constant for the heat-bath dynamics with product fixed point, we have mentioned that proving the existence of a positive log-Sobolev constant for a more general Lindbladian of the same form (sum of local terms) for any quantum channel with product fixed point is left as an open question.

Problem 12.5.9 For $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$, and $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$, prove, if true, that, if \mathcal{T}_x^* is a quantum channel with σ_x as fixed point for every $x \in \Lambda$, then

$$\sum_{x \in \Lambda} \mathcal{T}_x^* - \mathbb{1}_\Lambda$$

has a positive log-Sobolev constant.

Let us consider now the result on the log-Sobolev constant for the heat-bath dynamics, and more specifically the condition of strong quasi-factorization that appears in Assumption 10.3.2. Since it is left as a conjecture in a general case, there is a natural question concerning it, about the examples that might satisfy it.

Problem 12.5.10 Are there any easy examples of σ_Λ for which the strong quasi-factorization of Assumption 10.3.2 holds with f different from 1?

So far, the only example we have for this condition to hold is for σ_Λ a tensor product everywhere, for which the value of f is always 1. It is reasonable to think that this condition holds, for instance, when σ_Λ is a classical state, since in this case one could expect to recover the classical case, in which this inequality would agree with the usual quasi-factorization thanks to the DLR condition. However, this is left for future work.

Moreover, we can consider a similar question for the mixing condition appearing in Assumption 10.3.1.

Problem 12.5.11 Are there any more examples of σ_Λ for which the mixing condition of Assumption 10.3.1 holds?

Even though we have mentioned that this condition holds for classical states and we have shown a more complicated example of Gibbs state verifying this in Proposition 10.4.2, most of the tools available in the setting of quantum many-body systems to address the problem of decay of correlations on the Gibbs state depend strongly on the geometry used to split the lattice, and more specifically on the number of boundaries between the different regions A and B . Since, in our case, this number scales linearly with Λ , there is no hope to use any of those tools to obtain more examples of Gibbs states satisfying Assumption 10.3.1. However, it is possible that a different approach allows for more freedom in this sense.

Finally, also for the problem of positivity of the log-Sobolev constant for the heat-bath dynamics, we pose the following natural question.

Problem 12.5.12 Can we change the geometry used to split the lattice?

Another possible approach to tackle this problem could be based on the geometry presented in the classical papers [DPP02], [Ces01] and the quantum case for the spectral gap [KB16], such as the one used for Davies dynamics in Chapter 11. In this approach, in each step one splits the rectangle into two connected regions and carries out a more evolved geometric recursive argument. One of the main benefits from this approach would be a weakening in the mixing condition assumed in the Gibbs state. However, the main counterpart would be the necessity of a strong result of quasi-factorization for the relative entropy, even stronger than the one appearing in (10.28) (since the multiplicative error term should converge to 1 exponentially fast), in which both sides of the inequality would contain conditional relative entropies.



CONCLUSIONES Y PROBLEMAS ABIERTOS

En esta tesis, hemos estudiado evoluciones disipativas cuánticas y la velocidad de convergencia a su estado de equilibrio. Más específicamente, nos hemos planteado el problema de encontrar condiciones estáticas en el punto fijo de una evolución disipativa cuántica que sean condición suficiente para que la convergencia sea suficientemente rápida, esto es, que el sistema satisfaga equilibración rápida. Puesto que una constante de log-Sobolev proporciona una cota para el tiempo de equilibración de una evolución que podría escalar logarítmicamente con el tamaño del sistema bajo ciertas condiciones en el punto fijo, el problema de probar equilibración rápida se reduce a probar positividad de esta constante.

Por tanto, en esta tesis nos hemos centrado en encontrar condiciones estáticas en el punto fijo de una evolución disipativa cuántica que sean condición suficiente para que su generador tenga una constante de log-Sobolev positiva. El problema clásico análogo se estudió en el pasado siguiendo varias estrategias, de las que una de las más fructíferas utiliza un resultado de quasi-factorización de la entropía. Por tanto, primero nos hemos propuesto proporcionar un análogo cuántico a esta estrategia, basado en resultados de quasi-factorización de la entropía relativa.

Posteriormente, hemos aplicado la estrategia creada para las dinámicas de heat-bath y Davies bajo varias condiciones de agrupamiento de correlaciones en los puntos fijos de las evoluciones. Tras probar algunos resultados de quasi-factorización débil y fuerte a partir de algunas nociones de entropía relativa condicionada que hemos introducido previamente, los hemos utilizado para obtener resultados de positividad para las constantes de log-Sobolev asociadas a los generadores de heat-bath y Davies, respectivamente, bajo algunas condiciones de agrupamiento de correlaciones en los puntos fijos de las evoluciones.

Los resultados de esta tesis constituyen en su conjunto un gran paso hacia la solución del problema de probar positividad para constantes de log-Sobolev bajo ciertas condiciones en los puntos fijos de las evoluciones, puesto que no sólo hemos proporcionado resultados positivos para este problema para ciertas dinámicas, sino que también hemos creado una estrategia general que extiende a su análoga clásica y funciona al menos en los casos estudiados, dando lugar a nuevos interesantes resultados en esa línea.

This is an amazing sunset behind some palm trees in Venice Beach (California), during my visit to Caltech in April 2019.

Sin embargo, hay también muchos problemas naturales que surgen de esta tesis, tanto en la línea de los resultados de quasi-factorización como en la línea de las desigualdades de log-Sobolev. Enunciaremos algunos de ellos posteriormente, y mostraremos varios posibles enfoques para atacarlos.

Además, el campo de las desigualdades de log-Sobolev tiene (o podría tener) profundas implicaciones en muchas direcciones, no sólo para servir como condiciones suficientes para que un sistema cuántico satisfaga equilibración rápida. Algunas líneas de trabajo que la candidata podría continuar tras la compleción de esta tesis, tanto para extender resultados incluidos aquí como para aplicarlos (o las técnicas e ideas empleadas para obtenerlos) a diferentes escenarios serán presentados a continuación.

TRABAJO FUTURO

Las futuras líneas de investigación que surgen de esta tesis son numerosas. Por simplicidad, las podemos clasificar principalmente en dos clases: Extensiones de resultados que aparecen aquí o aplicaciones de dichos resultados a diferentes problemas.

Relativo a posibles extensiones de resultados de esta tesis, hay varias direcciones en las que podemos proceder ahora y que mencionamos a continuación.

- *Encontrar ejemplos de condiciones de agrupamiento de correlaciones.* Uno de los puntos más débiles de los resultados de positividad de constantes de log-Sobolev que aparecen en esta tesis es la falta de ejemplos para los que las condiciones de agrupamiento de correlaciones que asumimos se tengan. Por tanto, nos planteamos continuar el estudio de sistemas físicos para los cuales cualquiera de estas condiciones podría verificarse y probarlo. Un mejor entendimiento de las condiciones de agrupamiento asumidas para sistemas de espines clásicos podría ayudar en esta dirección.
- *Debilitar las condiciones suficientes.* Fuertemente relacionado con el punto anterior, un problema interesante que consideramos es debilitar las condiciones que asumimos para que la estrategia de probar positividad de la constante de log-Sobolev se cumpla y que aún seamos capaces de probar el resultado deseado. Si tuviésemos éxito, esto dejaría más espacio para que los sistemas cuánticos satisficieran las nuevas condiciones (posiblemente no tan fuertes).
- *Aumentar el tamaño de la clase de sistemas para los que nuestro resultado se cumple.* Otra línea de trabajo que seguimos ahora es intentar extender los resultados mencionados en este texto a clases más grandes. En particular, un resultado de este tipo sería la extensión de la positividad de la constante de log-Sobolev asociada al generador de heat-bath a sistemas con dimensión mayor que 1.
- *Entender las conexiones entre las distintas condiciones de agrupamiento de correlaciones.* Finalmente, entender mejor las conexiones entre las diferentes condiciones de agrupamiento de correlaciones conduciría a la existencia de más ejemplos de sistemas cuánticos para los que nuestro resultado se cumpla. En particular, pretendemos encontrar una relación entre la condición de equilibración asumida para la dinámica de heat-bath y el \mathbb{L}_1 -agrupamiento de correlaciones condicionado considerado para la dinámica de Davies, puesto que en el caso clásico ambas condiciones se reducen a la misma.

Desplacémonos ahora hacia las aplicaciones. Los resultados derivados en esta tesis, o las técnicas empleadas en ellos, se pueden utilizar para varios problemas interesantes de diversos campos, como mencionamos ahora.

- *Circuitos cuánticos.* Las constantes de log-Sobolev asociadas a la dinámica generada por

un canal cuántico que modela algún tipo de ruido (como el canal depolarizante) en un circuito cuántico se pueden utilizar para estimar la cantidad de ruido que el circuito puede presentar, y, por tanto, la fidelidad del circuito. Este enfoque nació bastante recientemente [BC19a], pero los resultados que se podrían obtener de ella parecen muy prometedores.

- *Ratios de equilibración de divergencias.* El procedimiento seguido para estimar el tiempo de equilibración de una evolución disipativa en función del ratio de equilibración dado por la entropía (la constante de log-Sobolev) se puede extender a situaciones más generales, como las generadas por las divergencias de Rényi o incluso las f -divergencias. Para ello, primero tendríamos que probar un resultado de quasi-factorización de la respectiva divergencia en función de unas condicionadas y posteriormente necesitaríamos refinar los resultados presentados en esta tesis a este nuevo escenario.
- *Capacidades cuánticas de canales.* La definición de la constante de log-Sobolev se deriva a partir de la entropía relativa. Si en lugar de ella, consideramos inicialmente otra cantidad, como la entropía condicionada, es fácil probar que la constante óptima correspondiente a la desigualdad funcional que es análoga a la desigualdad de log-Sobolev en el último caso proporciona una estimación a la capacidad cuántica de un canal [BCR19a], siguiendo los mismos pasos que para probar una cota superior en el tiempo de equilibración de una evolución disipativa en función de la constante de log-Sobolev. Este enfoque nos puede conducir a mejores estimaciones en capacidades de canales que los que ya se conocen.

PROBLEMAS ABIERTOS

Cambiamos ahora hacia los problemas que surgen de esta tesis. Una de las preguntas principales que aparece a partir del trabajo en entropías relativas condicionadas está relacionada con el resultado principal de la Sección 7.3.

En el Capítulo 9, demostramos que un resultado de quasi-factorización de la entropía relativa, cuando el segundo estado considerado es un producto tensor, es la pieza clave para probar que la dinámica de heat-bath, con punto fijo producto, tiene una constante de log-Sobolev positiva. La misma estrategia se podría seguir para obtener una constante de log-Sobolev positiva para la dinámica de heat-bath, en un ambiente más general, a partir del resultado de quasi-factorización presentado en el Teorema 7.3.1, bajo la suposición de un decaimiento de correlaciones en el punto fijo, como demostramos en el Capítulo 10. Sin embargo, es un problema abierto el saber si este resultado se puede utilizar con ese objetivo en dimensión mayor que 1.

Problema 12.5.13 *Usar el resultado de quasi-factorización de la entropía relativa en función de entropías relativas condicionadas de la Sección 7.3 para obtener constantes de log-Sobolev positivas para la dinámica de heat-bath en dimensión mayor que 1, con un punto fijo general σ .*

Considerando el enfoque seguido para la dinámica de Davies, es probable que, para conseguir un mejor resultado para heat-bath, necesitemos emplear las entropías relativas condicionadas por esperanzas en lugar de las usuales. Para ello, primero necesitamos mejorar el resultado de quasi-factorización para esta familia de entropías relativas condicionadas, Teorema 7.4.1.

Problema 12.5.14 *Mejorar el resultado de quasi-factorización de la entropía relativa condicionada por esperanzas del Teorema 7.4.1, a partir de mejorar la cota que obtenemos para el término de error.*

Relativo a la definición de entropía relativa condicionada presentada aquí, hemos mostrado varios detalles que nos permiten pensar que la definición es razonable. Sin embargo, hay algo de espacio para mejorarla, en algún sentido, de forma que podamos obtener resultados de quasi-factorización más fácilmente, por ejemplo.

Problema 12.5.15 *Mejorar la definición de entropía relativa condicionada. Una idea que se podría llevar a cabo es añadir la propiedad probada en Ecuación (7.1) a la definición.*

Más allá, ¿hay una posible definición axiomática para el concepto de entropía relativa condicionada a partir de la cual podamos obtener resultados de quasi-factorización inmediatamente?

Además, en la Sección 6.4, hemos comparado las definiciones de entropía relativa condicionada y entropía relativa condicionada por esperanzas. Por una parte, hemos demostrado que en varios casos coinciden, y por otra, que no siempre se puede dar la igualdad. Dejamos abierta la posibilidad de estudiar en general para qué casos ambas expresiones coinciden exactamente:

Problema 12.5.16 *Caracterizar para qué estados $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$, se cumple lo siguiente:*

$$D_A(\rho_{AB} || \sigma_{AB}) = D_A^E(\rho_{AB} || \sigma_{AB}),$$

o, al menos, encontrar ejemplos para que se tenga esta igualdad.

Posteriormente, al introducir el resultado de positividad de la constante de log-Sobolev para la dinámica de heat-bath con punto fijo producto tensor, hemos mencionado que probar la existencia de una constante de log-Sobolev positiva para un Lindbladiano más general de la misma forma (como suma de términos locales) para cualquier canal cuántico con punto fijo producto se deja como pregunta abierta.

Problema 12.5.17 *Para $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$, y $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$, probar, si es cierto, que, si \mathcal{T}_x^* es un canal cuántico con σ_x como punto fijo para cada $x \in \Lambda$, entonces*

$$\sum_{x \in \Lambda} \mathcal{T}_x^* - \mathbb{1}_\Lambda$$

tiene una constante de log-Sobolev positiva.

Consideremos ahora el resultado de la constante de log-Sobolev para la dinámica de heat-bath, y más específicamente, la condición de quasi-factorización fuerte que aparece en la Asunción 10.3.2. Puesto que se deja como una conjetura en el caso general, la pregunta más natural que surge de esta parte es encontrar ejemplos que la satisfagan.

Problema 12.5.18 *¿Hay ejemplos sencillos de σ_Λ para los que la quasi-factorización fuerte de la Asunción 10.3.2 se tenga con f diferente de 1?*

Hasta ahora, el único ejemplo que tenemos de que esta condición se cumpla es para σ_Λ un producto tensor en todas partes, para lo que el valor de f siempre es 1. Es razonable pensar que esta condición se tiene, por ejemplo, cuando σ_Λ es un estado clásico, puesto que en este caso uno esperaría poder reconstruir el caso clásico, en el que esta desigualdad coincidiría con la usual de quasi-factorización gracias a las condiciones DLR. Sin embargo, esto se deja para trabajo futuro.

Además, podemos considerar una cuestión similar para la condición de equilibración que aparece en la Asunción 10.3.1.

Problema 12.5.19 *¿Hay más ejemplos de σ_Λ para los que la condición de equilibración de la Asunción 10.3.1 se tenga?*

Aunque hemos mencionado que esta condición se tiene para estados clásicos y hemos mostrado un ejemplo más complicado de estado de Gibbs verificando esto en la Proposición 10.4.2, la mayoría de las herramientas disponibles en este escenario de sistemas cuánticos de muchos cuerpos para afrontar el problema del decaimiento de correlaciones en el estado de Gibbs dependen fuertemente de la geometría empleada para particionar la retícula, y más específicamente del número de fronteras entre las distintas regiones A y B . Puesto que, en nuestro caso, este número escala linealmente con Λ , no hay esperanza de que ninguna de esas

herramientas se pueda utilizar para obtener más ejemplos de estados de Gibbs que satisfagan la Asunción 10.3.1. Sin embargo, es posible que un enfoque diferente permita más libertad en este sentido.

Finalmente, también para el problema de positividad de la constante de log-Sobolev para la dinámica de heat-bath, nos planteamos la siguiente cuestión natural.

Problema 12.5.20 *¿Podemos cambiar la geometría empleada en la retícula?*

Otro posible enfoque para atacar este problema se podría basar en la geometría presentada en los artículos clásicos [DPP02], [Ces01] y el caso cuántico para el gap espectral [KB16], como la empleada para la dinámica de Davies en el Capítulo 11. En este enfoque, en cada paso se parte el rectángulo en dos regiones conexas y se lleva a cabo un argumento geométrico recursivo más elaborado. Uno de los principales beneficios de este enfoque vendría de debilitar la condición de equilibración asumida en el estado de Gibbs. Sin embargo, la principal contrapartida sería la necesidad de un resultado de quasi-factorización fuerte para la entropía relativa, incluso más fuerte que el que aparece en (10.28) (puesto que el término de error debería converger hacia 1 exponencialmente rápido), en el que en ambas partes de la desigualdad deberían aparecer entropías relativas condicionadas.



APPENDIX: NRA COMPACT OPERATORS

In this appendix, we briefly review another article obtained by the candidate and two coauthors during the period of her PhD in a completely different line of research (derived from the topic studied in her previous master thesis). We will not discuss this topic in detail and we just include here the main highlights of this project for completeness. The results mentioned below are based on [CMM17] and concern the problem of density of numerical radius attaining operators in a Banach space.

Let us begin by emphasizing that here we will consider, in general, infinite-dimensional Banach spaces (oppositely to the rest of the thesis, where we focus on finite-dimensional Hilbert spaces). We denote by X, Y or Z the Banach spaces that appear in this chapter, and given X a real or complex Banach space, we further denote by \mathbb{S}_X its unit sphere, and by \mathbb{B}_X its unit ball. Moreover, we denote by X^* the topological dual of X .

Given X and Y two Banach spaces, we write $\mathcal{B}(X, Y)$ for the space of bounded linear operators from X to Y and just $\mathcal{B}(X)$ whenever X coincides with Y . The space of compact linear operators on X is denoted by $\mathcal{K}(X)$.

Now, given an operator $T \in \mathcal{B}(X, Y)$, let us recall that its (operator) *norm* is given by

$$\|T\| := \sup_{x \in \mathbb{S}_X} \|Tx\|_Y,$$

where $\|\cdot\|_Y$ denotes the norm associated to the Banach space Y . We say that T *attains its norm* when the previous supremum is a maximum, i.e. whenever there exists $x \in \mathbb{S}_X$ such that $\|T\| = \|Tx\|_Y$. We write $NA(X, Y)$ for the set of *norm-attaining operators* between X and Y .

It is clear that, in general, not every bounded linear operator between two Banach spaces attains its norm, as it is easy to construct examples for which this happens even in separable Hilbert spaces. However, when the second space is the base field of the first one (which we denote by \mathbb{K}), and thus the set of bounded linear operators between them is just the set of functionals, Bishop and Phelps proved in [BP61] that the set of norm-attaining functionals $NA(X, \mathbb{K})$ is always dense in $\mathcal{B}(X, \mathbb{K})$ (for the topology given by the operator norm). Moreover, when X is finite-dimensional, it is easy to derive from the compactness of the unit ball that every functional attains its norm, and furthermore, if X is reflexive, this result also holds (indeed, it characterizes reflexivity of the space) and constitutes James Theorem.

This is a picture of the marvelous Alhambra, in Granada, where I studied my bachelor degree and where this project was started.

From these results arises the natural question of which conditions need to be imposed on two Banach spaces X and Y for $NA(X, Y)$ to be dense in $\mathcal{B}(X, Y)$. This opened a new field of study, which began with the seminal result of Lindenstrauss [Lin63], where he provided positive and negative examples for the aforementioned question. Afterwards, many results have been obtained in this field in the last half-century, and it is still an active field nowadays (a survey on results concerning norm-attaining operators can be found in [Cap15]). Again, if X is finite-dimensional, every operator from X to an arbitrary Y clearly attains its norm, but this does not hold anymore when X is reflexive unless we impose further conditions on Y .

Let us now consider the set of pairs of elements of X^* and X , both of norm 1, such that the element of X^* attains its norm at the element of X , that is

$$\Pi(X) := \{(x, x^*) \in X \times X^* : x \in \mathbb{S}_X, x^* \in \mathbb{S}_{X^*}, x^*(x) = 1\}.$$

Then, given an operator $T \in \mathcal{B}(X)$, the *numerical range* of T is the set of scalars given by

$$V(T) := \{x^*(Tx) : (x, x^*) \in \Pi(X)\},$$

and the *numerical radius* of T is obtained from this set as

$$v(T) := \sup\{|\lambda| : \lambda \in V(T)\}.$$

Note that, for every $T \in \mathcal{B}(X)$, we have $v(T) \leq \|T\|$. Analogously to the case of the norm, we say that an operator $T \in \mathcal{B}(X)$ *attains its numerical radius* if the previous supremum is a maximum and we denote the set of *numerical radius attaining operators* in X by $NRA(X)$. It is clear that every operator T attains its numerical radius if X is finite-dimensional, but it is also easy to construct examples in separable Hilbert spaces of operators which do not attain their numerical radii. The problem of the density of $NRA(X)$ was started by Sims (see [BS84], for instance) and arose parallelly to that of the density of $NA(X, Y)$. Some examples of spaces X such that

$$\overline{NRA(X)} = \mathcal{B}(X) \tag{12.23}$$

are Banach spaces with the Radon-Nikodym property [AP93] (in some sense, a version for vectorial measures of the Radon-Nikodym theorem), and $\mathbb{L}_1(\mu)$ spaces [Aco90].

Our paper [CMM17] concerns the negative version of this problem, i.e. for which Banach spaces X the set $NRA(X)$ is not dense in $\mathcal{B}(X)$. The first counterexample to Equation (12.23) was provided by Payá in [Pay92], shortly followed by another one obtained by Acosta, Aguirre and Payá [AAP92]. In these two examples, the operators shown that cannot be approximated by numerical radius attaining ones are not compact. Therefore, after the emergence of these results, the problem of finding a compact operator which cannot be approximated by numerical radius attaining ones was still open.

In [CMM17] we addressed this problem, namely we showed that there exists a Banach space X and a compact operator in X which cannot be approximated by numerical radius attaining ones. The analogous problem for norm-attaining operators had been recently solved by Martín [Mar14] and the strategy followed in our paper combined several ideas of the aforementioned [Pay92], [AAP92] and [Mar14].

The main result of [CMM17] reads as follows:

Theorem 12.5.21 — (CMM17).

There is a compact operator which cannot be approximated by numerical radius attaining operators.

Here we will just remark the key facts of the proof of this result. In a nutshell, inspired by the previous counterexamples for Equation (12.23), which had been previously inspired by [Lin63], we need to construct a space X from Y and Z such that Y verifies that Y^* is smooth enough, and Z fails to have extreme points in its unit ball in a strong way.

Moreover, since we want to construct a compact operator that cannot be approximated by numerical radius attaining operators, we need to further consider Y without the approximation property. Roughly speaking, recall that a Banach space X is said to have the *approximation property* if every compact operator can be approximated by finite-rank ones (the usual formulation of this property is in terms of compact sets, but this one is equivalent).

Next, for the condition mentioned above that we need Z to verify, let us define the following strong way of failing to have extreme points: We write

$$\text{Flat}(z_0) := \{z \in Z : \|z_0 \pm z\| \leq 1\},$$

and we say that Z is *strongly flat* if, for every $z_0 \in \mathbb{S}_Z$, the closed linear span of $\text{Flat}(z_0)$ has finite codimension. Note that, since z_0 is an extreme point of \mathbb{B}_Z if, and only if, $\text{Flat}(z_0) = \{0\}$, this condition fulfills the desired requirements.

Now, take into account the following facts:

1. For any $1 < p < 2$, the space ℓ_p has a closed subspace without the approximation property [LT79].
2. Both c_0 and all its closed infinite-dimensional subspaces are strongly flat [Mar16].
3. Let Y be such that the norm of Y^* is C^2 -smooth on $Y^* \setminus \{0\}$ and let Z be a strongly flat Banach space. Consider $X = Y \oplus_\infty Z$ and define $T \in \mathcal{B}(X)$ such that

$$T(y+z) = A(y) + B(z) \quad \forall y \in Y, z \in Z,$$

for $A \in \mathcal{B}(Y)$ and $B \in \mathcal{B}(Z, Y)$. If $T \in \text{NRA}(X)$, then B has finite rank [CMM17].

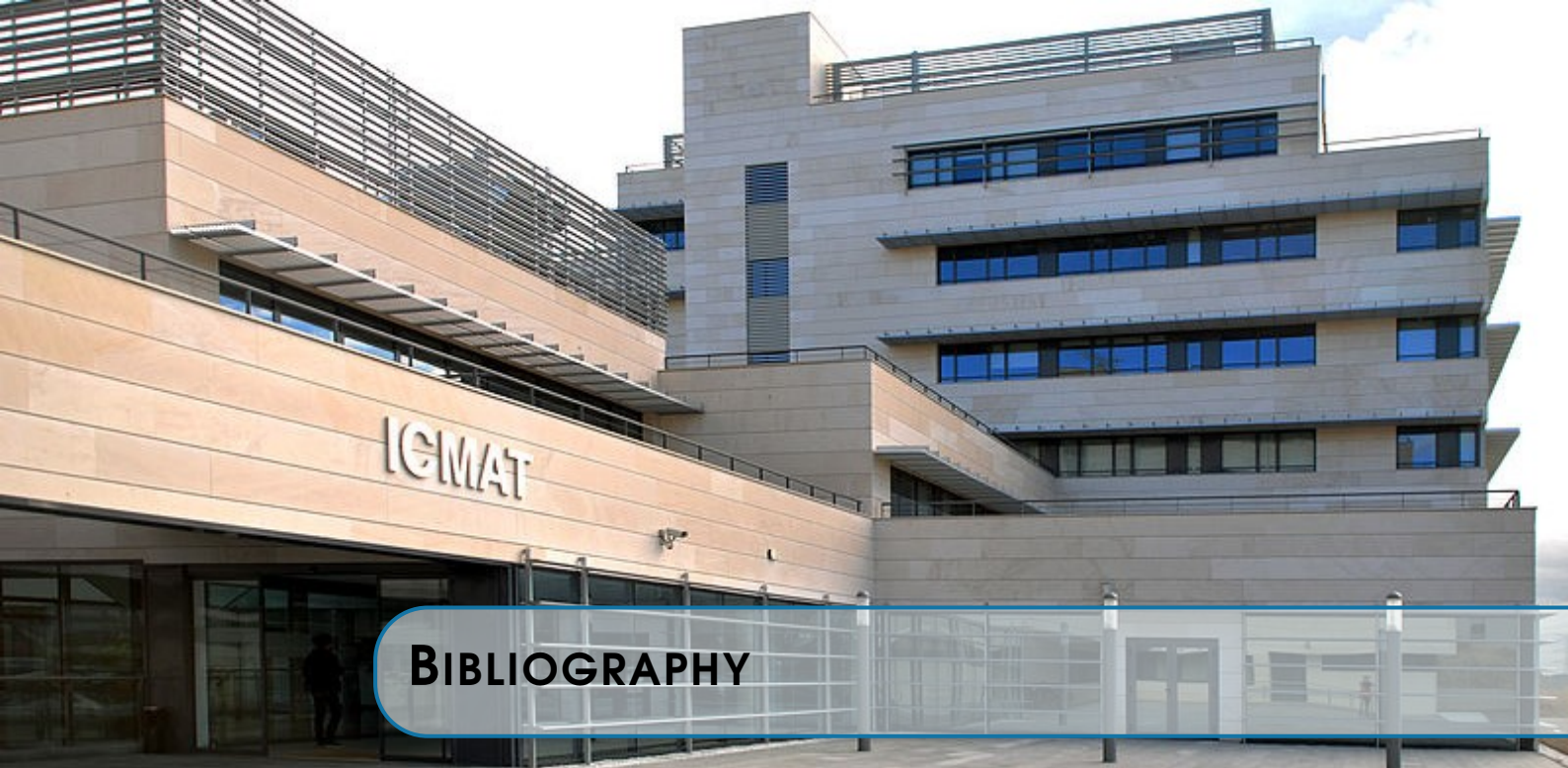
Note that, if we choose Y and Z verifying points (1) and (2) above, respectively, they are both in the conditions of (3). Hence, building on these ingredients, what we can actually prove is the following:

Given $1 < p < 2$, consider a subspace Y of ℓ_p without the approximation property. Then, there exists a closed subspace Z of c_0 such that:

$$\mathcal{K}(Y \oplus_\infty Z) \not\subseteq \overline{\text{NRA}(Y \oplus_\infty Z)}.$$

The rest of the proof of this result was shown in [CMM17], where we refer the reader for more information. Subsequently, to conclude the article, we showed some positive results on numerical radius attaining compact operators, that is examples of Banach spaces X for which

$$\overline{\text{NRA}(X) \cap \mathcal{K}(X)} = \mathcal{K}(X).$$



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This picture corresponds to the entrance of ICMAT (Instituto de Ciencias Matemáticas), the research center to which my fellowship was associated and in which I have spent most of the last four years.

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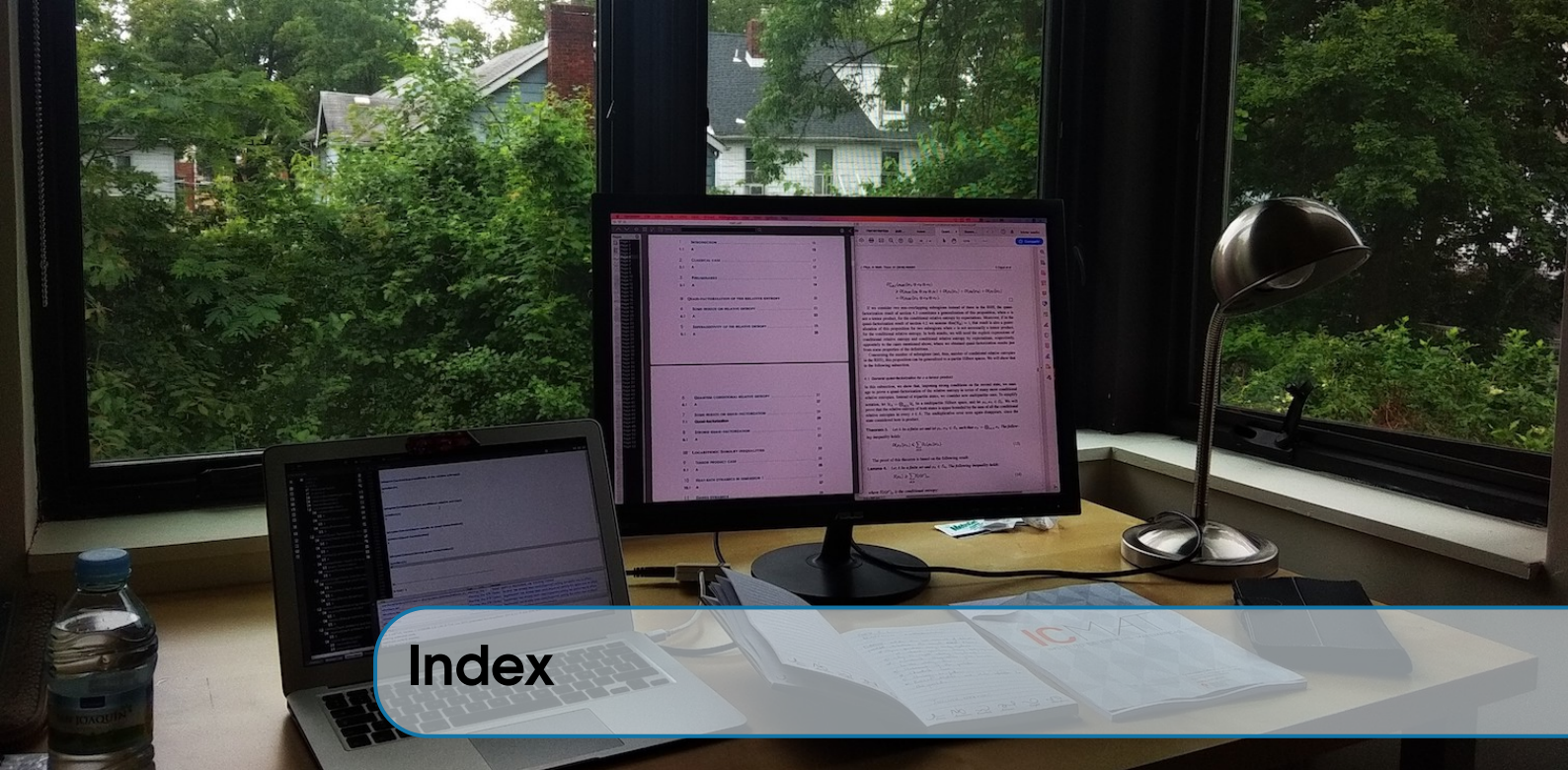
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Index

Symbols

f -divergence	
Maximal f -divergence	237
Standard f -divergence	236

B

Boundary condition	75
BS-entropy	238

C

Clustering condition	
Conditional \mathbb{L}_1 -clustering of correlations	161
Conditional \mathbb{L}_2 -clustering of correlations	169
Covariance-entropy clustering of correlations	172
Exponential conditional \mathbb{L}_1 -clustering of correlations	164
Exponential strong \mathbb{L}_2 -clustering of correlations	160
Conditional covariance	160
Conditional entropy	82
Conditional entropy production	103

Conditional expectation	96
Davies conditional expectation	215
Conditional Gibbs measure	76
Conditional log-Sobolev constant	103
Davies dynamics	217
Heat-bath dynamics	195
Tensor product	187
Conditional mutual information	92
Conditional relative entropy	
Conditional relative entropy	124
Conditional relative entropy by expectations	128
General conditional relative entropy by expectations	134
Conditional variance	160
Configuration space	74
Covariance	77
Covariance (quantum)	96

D

Decoherence-free subalgebra	216
Dirichlet form	78
DLR compatibility conditions	76
Dobrushin-Shlosman complete analyticity	82

E

Entropy	77
---------	----

Entropy constant 79
 Entropy production 102

F

Fidelity 232
 From log-Sobolev to conditional
 log-Sobolev constant for the
 Davies dynamics 219
 From log-Sobolev to conditional
 log-Sobolev constant for the
 Davies dynamics (2) 224

G

Gibbs measure 76
 Gibbs state 104
 Glauber dynamics 225

H

Hamiltonian 75, 104

I

Inequality
 General Hölder 94
 Golden-Thompson 113
 Hölder 94
 Jensen 99
 Lieb's extension of Golden-Thompson
 114
 Minkowski 94
 Multivariate trace inequalities'
 extension of Golden-Thompson
 114
 Pinsker 116
 Shearer 143
 Inner product
 σ -weighted inner product 30
 GNS inner product 95
 Hilbert-Schmidt inner product 30
 KMS inner product 95
 Ising model 77

K

KMS condition 215

L

Lattice and sites 73
 Lindbladian 100
 Detailed balance condition 101
 Local 103
 Primitive 101
 Local indistinguishability 121
 Local Topological Quantum Order 121
 Log-Sobolev constant 102
 Heat-bath dynamics in 1D 200
 Log-Sobolev constant for the Davies
 dynamics 225
 Tensor product case 189
 Logarithmic Sobolev 33
 Logarithmic Sobolev constant 79
 Lower asymptotically semicontinuity ... 93

M

Main results
 A condition for equality in the DPI for
 the BS entropy 240
 Axiomatic characterization of the
 conditional relative entropy .. 126
 From log-Sobolev to conditional
 log-Sobolev constant for the
 Davies dynamics 219
 From log-Sobolev to conditional
 log-Sobolev constant for the
 Davies dynamics (2) 224
 Log-Sobolev constant for the Davies
 dynamics 225
 Log-Sobolev constant for the heat-bath
 dynamics in 1D 200
 Log-Sobolev constant for the heat-bath
 for tensor products 189
 Quasi-factorization for quantum
 Markov chains 157
 Quasi-factorization for tensor product
 143

- Quasi-factorization for the conditional relative entropy 145
- Quasi-factorization for the conditional relative entropy by expectations 146
- Strengthened data processing inequality for the BS-entropy .246
- Strengthened DPI for maximal f -divergences for general channels 256
- Strong quasi-factorization for tensor product 156
- Strong quasi-factorization for the Cov-RE clustering of correlations 173
- Strong quasi-factorization under conditional \mathbb{L}_1 -clustering of correlations 165
- Superadditivity of the relative entropy for general states 112
- Mixing condition 199
 - Mixing condition 82
 - Strong mixing condition 81
 - Weak mixing condition 81
- Mixing time 101

N

- Norm
 - Schatten p -norm 94
 - Weighted norm 95

O

- Observables 29
- Operator convex 99
- Operator monotone 98

P

- Phase transition 77
- Poincaré 33
- Poisson measure 76
- Potential 75, 104
 - Commuting 104
 - Frustration-free 216
- Pseudo-conditional expectation 97

- Heat-bath conditional expectation .. 98
- Minimal conditional expectation ... 97

Q

- Quantum channel 30
- Quantum dissipative evolution 32
- Quantum Markov chain 104
- Quantum Markov semigroup 100
- Quasi-factorization 137
 - Conditional relative entropy 145
 - Conditional relative entropy by expectations 146
 - Non-Overlapping 138
 - Overlapping 137
 - Quantum Markov chains 157
 - Strong quasi-factorization for tensor product 156
 - Strong quasi-factorization under conditional \mathbb{L}_1 -clustering of correlations 165
 - Strong quasi-factorization under Cov-RE clustering of correlations 173
 - Tensor product 143

R

- Rényi divergence 236
- Rapid mixing 33
- Recovery map 104
- Rectangle and cube 74
- Relative entropy 77
- Relative entropy (quantum) 90

S

- Spectral gap 78
- States 29
- Stinespring's dilation theorem 100
- Strong quasi-factorization (assumption) 199
- Superoperator 30

T

- Total variation distance 77

V

Variance 77

Variance (quantum) 96

von Neumann entropy 90

W

Weak-coupling limit 31

To conclude this thesis, the picture of this chapter corresponds to the desk in Princeton where most of this thesis was written, during the summer of 2019.